49. On the Mean Value Property of Harmonic and Complex Polynomials

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1. Introduction. Throughout this note K denotes either the field of complex numbers C or the field of real numbers R. Let n be a fixed integer >2, and θ denote the number $\exp(2\pi i/n)$.

In 1935 S. Kakutani and M. Nagumo [1], and independently, in 1936 J. L. Walsh [3] proved the following theorems concerning the mean value property (MVP) of harmonic and complex polynomials.

Theorem A (Kakutani-Nagumo-Walsh). *If $f: C \to R$ is continuous, the* MVP

$$\sum_{\nu=0}^{n-1} f(x+\theta^{\nu} y)=n f(x)$$

holds for all $x, y \in C$ if, and only if, $f(x)$ is a harmonic polynomial of degree at most $n-1$.

Theorem B. *An entire function f satisfies the* MVP *for all $x, y \in C$ if and only if f is given by a complex polynomial of degree at most $n-1$.*

The above theorems are direct or indirect motivations for the generalizations and applications of various papers.

The main purpose of this note is to inform some more generalizations of Theorems A and B from the standpoint of the theory of finite difference functional equations.

2. The general solution. Definition. A mapping $Q^p: C \to K$ is called a homogeneous polynomial of degree p if and only if there exists a p-additive symmetrical mapping $Q_p: C^p \to K$; that is, $Q_p(x_1, \cdots, x_p) = Q_p(x_{i_1}, \cdots, x_{i_p})$ for all $x_1, \cdots, x_p \in C$ and for all permutations $(i_1, \cdots, i_p)$ of the sequence $(1, \cdots, p)$ and Q_p is an additive function in each x_q, $1 \leq q \leq p$, such that $Q^p(x)=Q_p(x, \cdots, x)$ for all $x \in C$. We say that Q_p is associated with Q^p or that Q_p generates Q^p.

We agree that for $p=0$ a homogeneous polynomial of degree zero is a constant.

Definition. Let β be any non-negative integer. If $f: C \to K$ is a finite sum $f=Q^0+Q^1+\cdots+Q^{\beta}$ of homogeneous polynomials, then f is called a generalized polynomial of degree at most β.

Notation. Let $Q_{(n-r, r)}(x ; y)$ denote the value of $Q_n(x_1, \cdots, x_n)$ for $x_i=x$, $i=1, \cdots, n-r$ and $x_i=y$, $i=n-r+1, \cdots, n$. In particular