30. On Eisenstein Series of Degree Two

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Introduction. We report arithmetical results on Eisenstein series of degree two. There exist two types of Eisenstein series of degree two. The first is the original Eisenstein series studied by Siegel [14] [15] and Maass [9] [10]. The second is the Eisenstein series attached to an elliptic cusp form constructed by Langlands [8] and Klingen [2]. We describe properties of Eisenstein series of degree two concerning the action of Hecke operators and the Fourier coefficients in a unified form. These results were motivated by [4] where certain arithmetical properties of Eisenstein series of degree two (of two types) were examined in connection with congruences between Siegel modular forms.

§ 1. Eisenstein series of degree two. We denote by $M_k(\Gamma_n)$ (resp. $S_k(\Gamma_n)$) the vector space over the complex number field $\boldsymbol{C}$ consisting of all Siegel modular (resp. cusp) forms of degree n and weight k for integers $n\geqq 0$ and $k\geqq 0$. We understand that $M_k(\Gamma_0)=S_k(\Gamma_0)=\boldsymbol{C}$ as usual. The space of Eisenstein series is denoted by $E_k(\Gamma_n)$ which is the orthogonal complement of $S_k(\Gamma_n)$ in $M_k(\Gamma_n)$ with respect to the Petersson inner product. Each modular form f in $M_k(\Gamma_n)$ has the Fourier expansion of the following form: $f=\sum_{T\geqq 0} a(T,f)q^T$ with $q^T=\exp(2\pi\sqrt{-1}\cdot\mathrm{trace}(TZ))$ where Z is a variable on the Siegel upper half space of degree n and T runs over all $n\times n$ symmetric semi-integral positive semi-definite matrices. For a subring R of $\boldsymbol{C}$ we put $M_k(\Gamma_n)_R=\{f\in M_k(\Gamma_n)\,|\,a(T,f)\in R$ for all $T\geqq 0\}$ (an R-module). We denote by $\mathrm{Aut}(\boldsymbol{C})$ the group of all field-automorphisms of $\boldsymbol{C}$. For $n\geqq 0$ and even $k\geqq 0$, each $\sigma\in\mathrm{Aut}(\boldsymbol{C})$ acts on $f=\sum_{T\geqq 0} a(T,f)q^T\in M_k(\Gamma_n)$ by $\sigma(f)=\sum_{T\geqq 0}\sigma(a(T,f))q^T\in M_k(\Gamma_n)$. We say that a modular form f in $M_k(\Gamma_n)$ is eigen if f is a non-zero eigenfunction of all Hecke operators on $M_k(\Gamma_n)$. We say that an eigen modular form f in $M_k(\Gamma_1)$ is normalized if $a(1,f)=1$. In this case we put $\boldsymbol{Q}(f)=\boldsymbol{Q}(\{a(n,f)\,|\,n\geqq 1\})$ and $\boldsymbol{Z}(f)=\boldsymbol{Q}(f)\cap\bar{\boldsymbol{Z}}$, where $\boldsymbol{Q}$ is the rational number field, $\boldsymbol{Z}$ is the rational integer ring, and $\bar{\boldsymbol{Z}}$ is the ring of all algebraic integers in $\boldsymbol{C}$. Note that $a(n,f)$ is the eigenvalue of the usual Hecke operator $T(n)$ for $f: T(n)f=a(n,f)f$. Then $\boldsymbol{Q}(f)$ is a totally real finite extension of $\boldsymbol{Q}$, and $\boldsymbol{Z}(f)$ is the integer ring of $\boldsymbol{Q}(f)$.

In this paper we restrict our attention to the space $E_k(\Gamma_2)$ of Eisenstein series of degree two. Hereafter in this section, k is an