25. *Class Number Calculation and Elliptic Unit. II*

Quartic Case

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Let K be a real quartic number field which is not totally real and contains a (real) quadratic subfield K_2. Let $D(<0)$, h and E_+ respectively be the discriminant, the class number and the group of positive units of K. In the following, an effective algorithm will be given to calculate h and E_+ at a time.

Our method is the same as in our preceding note [3] except for a slight change. We shall show a method to compute the relative class number with respect to K/K_2, assuming that the class number of K_2 is known.

§ 1. **Illustration of algorithm.** Let d_2, h' and $\eta_2\,(>1)$ respectively be the discriminant, the class number and the fundamental unit of K_2. We can compute h' and η_2 in a usual manner if d_2 is given. So we assume that h' and η_2 are explicitly given. The group E_+ of positive units of K is a free abelian group of rank 2. Let H_+ be the group of positive units of K/K_2, and $\varepsilon_1\,(>1)$ be the generator of H_+, i.e.

$$H_+ := \{\varepsilon \in E_+ \mid N_{K/K_2}(\varepsilon)=1\} = \langle \varepsilon_1 \rangle.$$

Then, as in [2], the relative unit ε_1 generates E_+ together with another unit $\varepsilon_2(>1)$, i.e. $E_+=\langle \varepsilon_1, \varepsilon_2\rangle$, where

$$(1) \qquad \varepsilon_2=\sqrt{\varepsilon_1\eta_2}, \quad \sqrt{\eta_2} \quad \text{or} \quad \eta_2.$$

Let η_e be the so-called "elliptic unit" of K, of which the definition will be given in § 5. Then, applying the results of Schertz [4], we see that $\eta_e>1$ and $\eta_e \in H_+$, and obtain the following relation between η_e and the class number h of K:

$$(2) \qquad h/h'=(E_+ : \langle \varepsilon_1, \eta_2\rangle)(H_+ : \langle \eta_e \rangle)/2.$$

Therefore, the calculation of the relative class number h/h' is reduced to the determination of the group index $(H_+ : \langle \eta_e\rangle)$ and the unit ε_2. Our method consists of the following steps:

(i) to compute an approximate value of η_e (§ 5),

(ii) to compute the minimal polynomial of η_e over $\boldsymbol{Q}$ (Lemma 2),

(iii) for $\xi \in H_+$ $(\xi>1)$, to give an explicit upper bound $B(\xi)$ of $(H_+ : \langle \xi \rangle)$ (Proposition 1),

(iv) for $\xi \in H_+$ $(\xi \neq 1)$, and for a natural number μ, to judge whether a real number $\sqrt[\mu]{\xi}$ belongs to K or not, and to compute the