64. The Basis Problem for Modular Forms on $\Gamma_0(N)$[†]

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(Communicated by Kunihiko KODAIRA, M. J. A., June 12, 1980)

§0. **Introduction.** Let $\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(Z) \mid c\equiv 0(N)\right\}$ and denote by $S_k(N, \psi)$ the space of cusp forms of weight $k\geqq 2$ and character ψ on $\Gamma_0(N)$. M. Eichler ([5, p. 77]) formulated the "Basis Problem", roughly speaking to "construct explicitly" a basis of $S_k(N, \psi)$, as a generalization of a conjecture of Hecke ([6, Satz 53]) and presented a solution in the case N is square free and $\psi=1$ ([3], [4], [5]). The purpose of this announcement is to sketch a "solution" for all weights $k\geqq 2$, all levels N, and all characters ψ mod N.

Let $S_k^0(N, \psi)$ denote the subspace of $S_k(N, \psi)$ generated by newforms. As it is easy to obtain a basis of $S_k(N, \psi)$ if one knows a basis of $S_k^0(m, \psi)$ for $m\mid N$, we restrict our attention to $S_k^0(N, \psi)$. Eichler's result has been generalized ([10], [14]) to yield: If N is not a square, $S_k^0(N, 1)$ is spanned by certain explicit theta series attached to quaternary quadratic forms associated to orders in (p, ∞)-quaternion algebras over $\boldsymbol{Q}$ (i.e. ramified at p and ∞), for various prime divisors p of N. If N is a square, such a result cannot hold in general. Using calculations of Parry [12], A. O. L. Atkin was able to discover in the case $S_2^0(13^2, 1)$ which newforms are not obtained from theta series and his questions and ideas about this to one of the present authors led to the "solution" for the case $S_k^0(p^2M, 1)$, p an odd prime, $p\nmid M$ ([15]).

Our general solution which includes all the above as special cases goes as follows. Call $S_k(N, \varphi)$ a *primitive neben space* if cond $(\varphi)=N$. An eigenform for the Hecke operators $T(n)$, $(n, N)=1$ in such a space will be called a *primitive nebenform*. Our main result shows how to explicitly decompose $S_k^0(N, \psi)$ as a direct sum of character twists of primitive neben spaces and twists of spaces spanned by certain "theta series" associated to (p, ∞)-quaternion algebras. That this is a reasonable solution to the basis problem follows from the result: For a newform f in $S_k^0(N, \psi)$ corresponding to the representation $\pi=\otimes\pi_\ell$ of the

†) Partially supported by NSF grant MCS-7703632-01.

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