61. *Polynomial Hamiltonians Associated with Painlevé Equations. I*[*)]

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1. Introduction. The purpose of this note is to study Hamiltonians associated with the six equations of Painlevé, and τ-functions related to Hamiltonians. The Painlevé equations are given by the following table:

$\mathrm{P_I}$ $\quad \lambda''=6\lambda^2+t$

$\mathrm{P_{II}}$ $\quad \lambda''=2\lambda^3+t\lambda+\alpha$

$\mathrm{P_{III}}$ $\quad \lambda''=\dfrac{1}{\lambda}(\lambda')^2-\dfrac{1}{t}\lambda'+\dfrac{1}{t}(\alpha\lambda^2+\beta)+\gamma\lambda^3+\dfrac{\delta}{\lambda}$

$\mathrm{P_{IV}}$ $\quad \lambda''=\dfrac{1}{2\lambda}(\lambda')^2+\dfrac{3}{2}\lambda^3+4t\lambda^2+2(t^2-\alpha)\lambda+\dfrac{\beta}{\lambda}$

$\mathrm{P_V}$ $\quad \lambda''=\left(\dfrac{1}{2\lambda}+\dfrac{1}{\lambda-1}\right)(\lambda')^2-\dfrac{1}{t}\lambda'+\dfrac{(\lambda-1)^2}{t^2}\left(\alpha\lambda+\dfrac{\beta}{\lambda}\right)+\dfrac{\gamma}{t}\lambda+\dfrac{\lambda(\lambda+1)}{\lambda-1}\delta$

$\mathrm{P_{VI}}$ $\quad \lambda''=\dfrac{1}{2}\left(\dfrac{1}{\lambda}+\dfrac{1}{\lambda-1}+\dfrac{1}{\lambda-1}\right)(\lambda')^2-\left(\dfrac{1}{t}+\dfrac{1}{t-1}+\dfrac{1}{\lambda-t}\right)\lambda'$

$$+\frac{\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[\alpha+\beta\frac{t}{\lambda^2}+\lambda\frac{t-1}{(\lambda-1)^2}+\delta\frac{t(t-1)}{(\lambda-t)^2}\right],$$

where α, β, γ and δ denote complex constants.

These equations are equivalent to the Hamiltonian systems

$$\begin{cases} \dfrac{d\lambda}{dt}=\dfrac{\partial \mathrm{H}}{\partial \mu} \\ \dfrac{d\mu}{dt}=-\dfrac{\partial \mathrm{H}}{\partial \lambda} \end{cases} \tag{1}$$

with a polynomial or rational Hamiltonian $\mathrm{H}=\mathrm{H}(t;\lambda,\mu)$. Historically this fact was first remarked by J. Malmquist, in his paper studying polynomial systems of differential equations without movable branch points and some explicit forms of polynomial Hamiltonians were given for the Painlevé equations except for the third one ([1], p. 86). Recently, the author showed that the Painlevé equations are equivalent to systems of the form (1) and gave each system a geometric interpretation ([2], p. 47).

2. Isomonodromic deformations. Consider firstly the linear differential equation

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