18. *Euler's Finite Difference Scheme and Chaos*

By Masaya YAMAGUTI and Hiroshi MATANO
Kyoto University

(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1979)

1. Introduction. Despite the simple dynamical structure of scalar ordinary differential equations, the corresponding difference equations (Euler's scheme) sometimes exhibit a very complicated dynamical behavior. Such phenomena have recently been understood somewhat systematically from the viewpoint of "chaos" theory, which is in course of powerful development since the work of Li-Yorke [1]. In this short note we shall make clear the importance of the role which asymptotically stable equilibrium points play in the process of chaotic phenomena. In other words, it will be shown that a best-settled equilibrium point in the original differential equation is actually apt to turn into a source of chaos in the corresponding difference equation. Our research here was inspired by R. M. May's example of $\frac{du}{dt}=u(\varepsilon-hu)$.

2. Notation and theorem. Let us consider scalar differential equations of the form

$$\frac{du}{dt}=f(u), \tag{1}$$

where $f(u)$ is continuous in $\boldsymbol{R}^1$. We assume that (1) has at least two equilibrium points one of which is asymptotically stable. As is easily seen, this assumption reduces (after a linear transformation of the unknown if necessary) to the conditions

(*) $\quad f(0)=f(\bar{u})=0 \quad$ for some $\bar{u}>0$,

(*) $\quad f(u)>0 \quad (0<u<\bar{u})$,

(*) $\quad f(u)<0 \quad (\bar{u}<u<K)$.

Here the constant K is possibly $+\infty$. Euler's difference scheme for (1) takes the form

$$x_{n+1}=x_n+\Delta t\cdot f(x_n), \tag{2}$$

and henceforth we will adopt the notation $F_{\Delta t}(x)=x+\Delta t f(x)$. Our theorem can now be stated as follows:

Theorem. i) *Let* (*) *hold. Then there exists a positive constant* c_1 *such that for any* $\Delta t>c_1$ *the difference equation* (2) *is chaotic in the sense of Li-Yorke.*

ii) *Suppose in addition that* $K=+\infty$; *then there exists another*