74. *A Note on Modular Forms mod $\mathfrak{p}$*

By Masao KOIKE
Department of Mathematics, Faculty of Sciences,
Nagoya University

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Let N be a positive integer and χ be a Dirichlet character mod N. Let $\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in SL_2(\boldsymbol{Z}) \mid c\equiv 0 \pmod N\right\}$. Let $f(z)$ be a cusp form of weight k satisfying

$$f(\sigma(z))=(cz+d)^k\chi(d)f(z) \qquad \text{for all } \sigma=\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(N).$$

Then we call $f(z)$ a cusp form of type (k,χ) on $\Gamma_0(N)$, and we denote by $S_k(N,\chi)$ the space of all cusp forms of type (k,χ) on $\Gamma_0(N)$.

From now we fix a prime number $p, p\geq 5$. Let N be a positive integer such that $(p,N)=1$. Let ψ and χ be any Dirichlet characters mod N and mod p respectively such that $\psi\chi(-1)=1$. Let t be the order of χ and put $\kappa=\dfrac{(p-1)(t-a)}{t}$ with an integer a such that $1\leq a\leq t$ and $(a,t)=1$. Let k be any even positive integer. Then we can prove the following simple identities between dimensions of spaces of cusp forms by using Hijikata's trace formula [1]:

Theorem 1. *The notations being as above, we have*

$$\dim_C S_k(Np,\psi\chi)=\dim_C S_{(p+1)k/2-\kappa}(N,\psi)+\dim_C S_{(p+1)k/2-(p-1-\kappa)}(N,\psi).$$

As an application of Theorem 1, we can study some properties of cusp forms mod p in the sense of Serre and Swinnerton-Dyer.

We fix our notations. We may fix N, ψ and k. Take an algebraic number field K of finite degree over the rational number field which contains all eigenvalues of all Hecke operators acting on $S_k(Np,\psi\chi)$ for all Dirichlet characters χ mod p and on $S_{k'}(N,\psi)$ for all $k'\leq \dfrac{k}{2}(p+1)$, and p-th roots of unity. We fix a prime divisor $\mathfrak{p}$ of K lying over p. Let ν be the normalized valuation of K attached to $\mathfrak{p}$ so that $\nu(p)=p^{-1}$ and $\mathfrak{o}=\{\alpha\in K \mid \nu(\alpha)\leq 1\}$, $F=\mathfrak{o}/\mathfrak{p}$.

For any Dirichlet character χ mod p, let

$$V_\chi=\left\{f(z)=\sum_{n=1}^{\infty} a_n q^n \in S_k(Np,\psi\chi) \mid a_n\in K \quad \text{for all } n\geq 1\right\},$$

$$V_{k'}=\left\{g(z)=\sum_{n=1}^{\infty} b_n q^n \in S_{k'}(N,\psi) \mid b_n\in K \quad \text{for all } n\geq 1\right\},$$

where $q=e^{2\pi iz}$. V_χ and $V_{k'}$ are vector spaces over K with same dimensions as those of $S_k(Np,\psi\chi)$ and $S_{k'}(N,\psi)$ over the complex number field.