59. *Isomorphism Criterion and Structure Group Description for $\mathfrak{N}$-Semigroups*

By James M. LORD
University of California, Davis

(Communicated by Kôsaku YOSIDA, M. J. A., Sept. 12, 1979)

0. Introduction. A commutative cancellative archimedean semigroup without idempotents is called an $\mathfrak{N}$-semigroup. In this paper, necessary and sufficient conditions are given for two $\mathfrak{N}$-semigroups to be isomorphic and the structure groups of an $\mathfrak{N}$-semigroup are completely described. M. Sasaki did some related work in [1], but the results given here are simpler. In [2], T. Tamura obtained an isomorphism criterion from a different point of view.

1. Preliminaries. Let S be any $\mathfrak{N}$-semigroup and let $a\in S$. Define a group-congruence ρ_a on S by $x\rho_a y$ if and only if $a^m x=a^n y$ for some $m,n\in Z_+$ (the positive integers). The group $G_a=S/\rho_a$ is called the structure group of S with respect to a. Structure group products will be denoted by "$*$" in this paper. Let $p\in S$. If $p\notin aS$, p is called a prime (relative to a). Every $x\in S$ has a unique representation $x=a^k p$ where $k\in Z_+^0$ ($a^0 p$ means p) and $p\in S$ is a prime. By the fundamental structure theorems for $\mathfrak{N}$-semigroups [2], we may assume $S=(G;I)=(G;\varphi)$, that is, $S=\{(x,\xi):x\in Z_+^0,\xi\in G\}$ where $(x,\xi)(y,\eta)=(x+y+I(\xi,\eta),\xi*\eta)$ and $I(\xi,\eta)=\varphi(\xi)+\varphi(\eta)-\varphi(\xi*\eta)$ for all $\xi,\eta\in G$. Let $(m,\alpha)\in S$. The structure group $G_{(m,\alpha)}=S/\rho$ is of major importance in this paper. Observe that $G_{(m,\alpha)}=\{\overline{(x,\xi)}:(x,\xi)$ is prime relative to $(m,\alpha)\}$. For a more thorough review of $\mathfrak{N}$-semigroups, see [2].

2. Isomorphism criterion. Theorem 2.1. *Let $S=(G;I)=(G;\varphi)$ and $\hat{S}=(\hat{G};\hat{I})=(\hat{G};\hat{\varphi})$. Then S is isomorphic to $\hat{S}$ if and only if there exists $(m,\alpha)\in S$ such that*

(2.1.1) *$G_{(m,\alpha)}$ is isomorphic to $\hat{G}$ and*

(2.1.2) $\hat{\varphi}(\hat{\xi})+\hat{\varphi}(\hat{\eta})-\hat{\varphi}(\hat{\xi}*\hat{\eta})=(x+\varphi(\xi)+y+\varphi(\eta)-(z+\varphi(\gamma)))/(m+\varphi(\alpha))$ *holds for all $\hat{\xi},\hat{\eta}\in\hat{G}$ where (x,ξ), (y,η), and (z,γ) are the unique primes in S relative to (m,α) such that the isomorphism in (2.1.1) carries $\overline{(x,\xi)}$, $\overline{(y,\eta)}$, and $\overline{(z,\gamma)}$ to $\hat{\xi}$, $\hat{\eta}$, and $\hat{\xi}*\hat{\eta}$ respectively.*

Proof. *Necessity.* Assume $f:S\to\hat{S}$ is the isomorphism and let $f(m,\alpha)=(0,\hat{\varepsilon})$. Define $\iota:\hat{G}_{(0,\hat{\varepsilon})}\to\hat{G}$ by $\iota\overline{(0,\hat{\xi})}=\hat{\xi}$ and $\hat{f}:G_{(m,\alpha)}\to\hat{G}_{(0,\hat{\varepsilon})}$ by $\hat{f}\overline{(x,\xi)}=\overline{f(x,\xi)}$. Then $\iota\circ\hat{f}$ is an isomorphism of $G_{(m,\alpha)}$ onto $\hat{G}$. To prove (2.1.2), let $\hat{\xi},\hat{\eta}\in\hat{G}$ and let (x,ξ), (y,η), and (z,γ) be the primes relative to (m,α) such that $(\iota\circ\hat{f})\overline{(x,\xi)}=\hat{\xi}$, $(\iota\circ\hat{f})\overline{(y,\eta)}=\hat{\eta}$, and $(\iota\circ\hat{f})\overline{(z,\gamma)}=\hat{\xi}*\hat{\eta}$. Then $f(x,\xi)=(0,\hat{\xi})$, $f(y,\eta)=(0,\hat{\eta})$, and $f(z,\gamma)=(0,\hat{\xi}*\hat{\eta})$. Define