58. Identities E-2 and Exponentiality

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1. Introduction. Let S be a semigroup and Z_+ the set of positive integers, and $Z_+^0=Z_+\cup\{0\}$. Define $E(S)$ by

$$E(S)=\{n\in Z_+ : (ab)^n=a^nb^n \quad \text{for all } a, b\in S\}.$$

$E(S)$ is a multiplicative semigroup containing 1. A semigroup is called an E-n semigroup [3] if $n\in E(S)$. If $E(S)=Z_+$, then S is called *exponential*. In some semigroups, E-2 implies exponentiality, for example, this holds for groups, cancellative semigroups or inverse semigroups [4]. More generally, regular E-2 semigroups are exponential [3]. Recently A. Cherubini Spoletini and A. Varisco [1] obtained that a power cancellative E-2 semigroup is exponential and also they have

Proposition 1 ([1]). *Let S be a semigroup. If $n\in E(S)$ then $n+\lambda n(n-1)\in E(S)$ for all $\lambda\in Z_+^0$. Hence if $2\in E(S)$ then $2n\in E(S)$ for all $n\in Z_+$.*

As is well known [2], if S is a group and $E(S)$ contains three consecutive integers then $E(S)=Z_+$. In parallel to this,

Proposition 2 ([5]). *Let S be a semigroup. If 2, n, $n+1$ and $n+2$ are in $E(S)$ then $\{m\in Z_+ : m\geq n\}\subset E(S)$. Therefore, if S is E-2 and E-3, then S is exponential.*

In this paper, Theorem 3 will improve Proposition 2 and the second part of Proposition 1 so that we shall be able to completely describe $E(S)$ when $2\in E(S)$.

2. Results. Theorem 3. *Let S be a semigroup. If $2\in E(S)$, then $m\in E(S)$ for all $m\geq 4$.*

Proof. Since $2\in E(S)$ implies $4\in E(S)$, it is sufficient to verify the following: If $n>2$ and 2, $n\in E(S)$, then $n+1\in E(S)$.

In case n is odd, $n-1$ is even, so let $n-1=2k$. Then

$$\begin{aligned} x^{n+1}y^{n+1} &= x(x^ny^n)y=x(xy)^ny=x^2(yx)^{n-1}y^2 \\ &= x^2((yx)^k)^2y^2=(x(yx)^k)^2y^2=((xy)^kx)^2y^2 \\ &= (xy)^{2k}x^2y^2=(xy)^{n-1}(xy)^2=(xy)^{n+1}. \end{aligned}$$

In case n is even, let $n-2=2k$. Then

$$\begin{aligned} x^{n+1}y^{n+1} &= x(x^ny^n)y=x(xy)^ny=x^2(yx)^{n-1}y^2 \\ &= x^2(yx)^{n-3}(yx)^2y^2=x^2(yx)^{n-3}(yxy)^2 \\ &= x^2(yx)^{n-2}y^2xy=x^2((yx)^k)^2y^2xy \\ &= (x(yx)^k)^2y^2xy=((xy)^kx)^2y^2xy=(xy)^{2k}x^2y^2xy \\ &= (xy)^{n-2}(xy)^2(xy)=(xy)^{n+1}. \end{aligned}$$