52. A Note on Almost-Primes in Short Intervals

By Yoichi MOTOHASHI
Department of Mathematics, College of Science
and Technology, Nihon University

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1. In this note we are concerned with the existence of P_2 (numbers having at most two prime factors) in almost all short intervals. The hitherto best result in this field is due to Heath-Brown [1], who has shown, using the weighted linear sieve, that for almost all x there exists a P_2 such that $x<P_2\leqq x+x^{1/11}$. Improving on this we shall prove, by an easy variant of Jutila's argument [2], the following result:

Theorem. *Let ε be an arbitrary small positive constant. Then for almost all x there exists a P_2 such that $x<P_2\leqq x+x^{\varepsilon}$.*

Before giving the proof we make some remarks. One may consider the possibility of reducing the length of intervals from x^{ε} to a power of $\log x$, which is desirable especially when Selberg's result [4] on primes in short intervals is taken into account. But the best result our argument can yield seems to be only $\exp(C(\log x)^{2/3}(\log\log x)^{4/3})$ with a large constant C, instead of x^{ε} in our theorem. Also one may ask how the situation is if we consider the least P_2 in almost all arithmetic progressions modulo a fixed integer. Because of the lack of a result on Dirichlet's L-functions comparable with Vinogradov's zero-free region for $\zeta(s)$ the Riemann zeta-function, our argument in the present note cannot be modified so as to be applicable to arithmetic progressions. Thus our result in [3] remains so far to be the best.

2. Now we enter into the proof. Let x be sufficiently large, and let us assume that $x>U, V>x^{1/3}$. And we put, denoting primes by p, p',

$$P(s)=\sum_{U<p\leqq 2U} p^{-s},$$

$$\Phi(y)=\sum_{\substack{y<pp'\leqq y(1+1/V)\\ U<p\leqq 2U}} \log p'.$$

Further we put

$$I_x=\frac{1}{x}\int_x^{2x}\left|\Phi(y)-\frac{y}{V}P(1)\right|^2 dy.$$

We have

$$\Phi(y)-\frac{y}{V}P(1)$$