45. On a Nature of Convergence of Some Feynman Path Integrals. I

By Daisuke FUJIWARA
Department of Mathematics, University of Tokyo

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§1. **Introduction.** In the previous papers [6], [7] and [8], we discussed convergence in the uniform operator topology of the Feynman path integral under some assumptions concerning the potential function. In this note, we shall discuss pointwise convergence of the Feynman path integral and we shall prove that it converges in a very strong topology if the potential function satisfies the same assumptions as in the previous papers [6], [7] and [8]. As to the notion of the Feynman path integral we refer Feynman [4] and Feynman-Hibbs [5].

Let $x=(x_1, x_2, \cdots, x_n)$ denote a point of $\mathbf{R}^n$. We shall treat the quantum dynamical system described by the Lagrangean of the form

$$L(t, x, \dot{x})=\frac{1}{2}|\dot{x}|^2-V(t, x). \tag{1}$$

The potential function $V(t, x)$ is assumed to satisfy the following assumptions;

(A-I) *$V(t, x)$ is a real-valued function of $(t, x)\in \mathbf{R}\times\mathbf{R}^n$. For any fixed $t\in\mathbf{R}$, $V(t, x)$ is a function of $x\in\mathbf{R}^n$ of class C^∞. $V(t, x)$ is a measurable function of $(t, x)\in\mathbf{R}\times\mathbf{R}^n$.*

(A-II) *For any multi-index α with its length $|\alpha|\geq 2$, the nonnegative measurable function of t defined by*

$$M_\alpha(t)=\sup_{x\in\mathbf{R}^n}\left|\left(\frac{\partial}{\partial x}\right)^\alpha V(t, x)\right|+\sup_{|x|\leq 1}|V(t, x)| \tag{2}$$

is essentially bounded on every compact interval of $\mathbf{R}$.

We fix a large integer K, say, $K=100(n+100)$. We put $T=\infty$ if $\sum_{2\leq|\alpha|\leq K}\text{ess. sup}_{t\in\mathbf{R}} M_\alpha(t)<\infty$. Otherwise, we let T denote an arbitrary fixed positive number. We shall discuss everything in the time interval $(-T, T)$.

Let $S(t, s, x, y)$ be the classical action along the classical orbit starting from y at time s and reaching x at time t. We can prove that there exists a positive constant $\delta_1(T)$ such that $S(t, s, x, y)$ is uniquely defined for any x and y in $\mathbf{R}^n$ if $|t-s|\leq\delta_1(T)$(cf. [4, Proposition 1]).

We shall consider the following integral transformation:

$$E^{(0)}(\lambda, t, s)\varphi(x)=\left(\frac{-\lambda}{2\pi(t-s)}\right)^{n/2}\int_{\mathbf{R}^n} e^{\lambda S(t,s,x,y)}\varphi(y)dy, \tag{3}$$

where $\lambda=i\hbar^{-1}$, $\hbar$ being a small positive parameter (=the Planck con-