43. A Note on Siegel's Zeros

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1. Let χ be a real primitive Dirichlet character (mod q), and $L(s,\chi)$ the L-function attached to χ. Then, it may happen that $L(s,\chi)$ has a real zero $1-\delta$ such that

$$0<\delta\leqq c_1(\log q)^{-1},$$

where and also in the sequel c's are absolute constants. This, if exists, is the Siegel zero of $L(s,\chi)$.

On the other hand, let $\pi(x;q,l)$ be as usual the number of primes less than x and congruent to l (mod q). Then, as is well-known, the hypothetical estimate

$$\pi(x;q,l)\leqq(2-\xi)\frac{x}{\varphi(q)\log x/q}, \tag{1}$$

where $\xi>0$ is an absolute constant, and $\varphi(q)$ is the Euler function, implies

$$\delta\geqq c_2\xi(\log q\log\log q)^{-1} \tag{2}$$

(cf. [3]). It seems that as far as we appeal to the prime number theorem of Rodosskii and Tatuzawa [6, p. 314] the above result (2) is the best that can be deduced from (1).

2. The purpose of this short paper is to show the result which appears to be ultimately the best possible one deducible from (1):

Theorem. *If* (1) *holds for* $x\geqq q^{c_3}$, *then we have*

$$\delta\geqq c_4\xi(\log q)^{-1}.$$

Proof. There are two ways to prove this. One is via the prime number theorem of Linnik-Fogels-Gallagher [2] (see also [4]). The other one, which we are going to show below, is closely related to the Deuring-Heilbronn phenomenon,*) and much more elementary and direct.

Now, let us put

$$B(n)=\sum_{d|n}\chi(d)d^{-\delta}$$

which is non-negative for all n. And let us apply the Selberg sieve to the sequence $\{B(n)\}$. That is, we consider the expression

$$I(N,z)=\sum_{n\leqq N}B(n)\Big(\sum_{d|n}\lambda_d\Big)^2,$$

*) This fact will be analysed in our forthcoming paper in a wider context including large sieve extensions of (1) (cf. [1] [5]).