# 31. On the Unique Maximal Idempotent Ideals of Non-Idempotent Multiplication Rings

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In the preceding paper [5], we have defined multiplication rings, shortly $M$-rings, as rings s.t. for any ideals $\mathfrak{a}, \mathfrak{b}$, with $\mathfrak{a}<\mathfrak{b}$, there exist ideals $\mathfrak{c}, \mathfrak{c}'$, s.t. $\mathfrak{a}=\mathfrak{b}\mathfrak{c}=\mathfrak{c}'\mathfrak{b}$; here "$<$" means a proper inclusion. An $M$-ring is called non-idempotent, if $R>R^2$. We have proved that the unique maximal idempotent ideal $\mathfrak{d}$ of a non-idempotent $M$-ring can be obtained as an intersection of some ideal sequence $\{\mathfrak{d}_\alpha\}_\wedge$, where $\mathfrak{d}_\alpha$ are defined inductively ([5], Theorem 5): $\mathfrak{d}=\bigcap_{\alpha\in\wedge}\mathfrak{d}_\alpha$. In § 1, we shall prove that $\mathfrak{d}$ is an essential submodule of $R$, both as a left and also as a right $R$-module, and at the end of the section we shall give an example of a non-idempotent $M$-ring with $\mathfrak{d}\neq\{0\}$. If moreover $R$ is left Noetherian, and let $N$ denote the Jacobson radical of $R$, then by Theorem 5 (i) [5], $N\subseteq\mathfrak{d}$ or $N=\mathfrak{d}_\alpha^j$ for some ordinal $\alpha$ and some positive integer $j$. If $N=\mathfrak{d}$ or $N=\mathfrak{d}_\alpha^j$, then by Theorem 5 (ii) [5] and Nakayama's lemma $\mathfrak{d}=\{0\}$, so we have to consider the case $N<\mathfrak{d}$ only; so in § 2 we consider left Noetherian non-idempotent $M$-rings, and prove that any ideal, which is maximal in the set of ideals properly contained in $\mathfrak{d}$, is a prime ideal of $R$.

**1. Non-idempotent M-rings.** **Lemma 1.** *Let $R$ be a non-idempotent M-ring, and let $\mathfrak{a}$ be any ideal, s.t. $\mathfrak{a}\subseteq\mathfrak{d}$ then $\mathfrak{d}\mathfrak{a}=\mathfrak{a}\mathfrak{d}=\mathfrak{a}$; furthermore for an ideal $\mathfrak{d}'$ s.t. $\mathfrak{d}\subseteq\mathfrak{d}'$, $\mathfrak{a}\mathfrak{d}'=\mathfrak{d}'\mathfrak{a}=\mathfrak{a}$.*

**Proof.** If $\mathfrak{a}=\mathfrak{d}$, there is nothing to prove. If $\mathfrak{a}<\mathfrak{d}$, then $\mathfrak{a}=\mathfrak{d}\mathfrak{b}=\mathfrak{b}'\mathfrak{d}$ for some ideals $\mathfrak{b}, \mathfrak{b}'$, therefore $\mathfrak{a}\mathfrak{d}=\mathfrak{b}'\mathfrak{d}\cdot\mathfrak{d}=\mathfrak{b}'\mathfrak{d}=\mathfrak{a}$. Similarly $\mathfrak{d}\mathfrak{a}=\mathfrak{a}$.

**Lemma 2.** *Let $R$ be a non-idempotent M-ring, and let $N<\mathfrak{d}$, then $N=\bigcap_{I\in\mathfrak{M}}I=\bigcap_{J\in\mathfrak{N}}J$, where $\mathfrak{M}$ and $\mathfrak{N}$ denote the set of maximal left ideals of $R$, and all maximal right ideals of $R$ respectively.*

**Proof.** In general, $NR\subseteq\bigcap_{I\in\mathfrak{M}}I\subseteq N$, and $\bigcap_{I\in\mathfrak{M}}I$ is an ideal of $R$. By Lemma 1 $N=NR$, hence equality holds.

**Theorem 1.** *Let $R$ be a non-idempotent M-ring. If $R\neq N$, then $N=\bigcap_{I\in\mathfrak{M}}I=\bigcap_{J\in\mathfrak{N}}J$, where $\mathfrak{M}, \mathfrak{N}$ is the same as Lemma 2.*

**Proof.** By Proposition 4 [5], $N=R$ or $N\subseteq\mathfrak{d}$. If $N=\mathfrak{d}$, then $\mathfrak{d}=\mathfrak{d}R=NR\subseteq\bigcap_{I\in\mathfrak{M}}I\subseteq\mathfrak{d}$, therefore $\mathfrak{d}=N=\bigcap_{I\in\mathfrak{M}}I$. If $N<\mathfrak{d}$, the results follow by Lemma 2.

**Lemma 3.** *Let $R$ be a non-idempotent M-ring, and let $I$ be any maximal left ideal of $R$, then $I\mathfrak{d}=\mathfrak{d}$. The similar results hold for right*