32. *Cosine Families and Weak Solutions of Second Order Differential Equations*

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Let A be a densely defined closed linear operator on a real or complex Banach space X, let $T>0$ and let $g \in L^1(0, T; X)$. Recently, Ball [1] proved that there exists for each $x \in X$ a unique weak solution, suitable defined, of the equation $u'(t)=Au(t)+g(t)$, $t \in [0, T]$, $u(0)=x$ if and only if A is the infinitesimal generator of a (C_0)-semigroup $\{T(t); t \geq 0\}$ on X, and in this case the solution $u(t)$ is given by

$$u(t)=T(t)x+\int_0^t T(t-s)g(s)ds, \qquad t \in [0, T].$$

The purpose of this note is to establish the parallel relationship between cosine families and second order differential equations

$$(\mathrm{IV}; x, y) \qquad \begin{cases} w''(t)=Aw(t)+g(t), & t \in [0, T], \\ w(0)=x \in X, & w'(0)=y \in X. \end{cases}$$

Let A^* denote the adjoint of A and $\langle\ ,\ \rangle$ the pairing between X and its dual space X^*.

Definition. A function $w \in C([0, T]; X)$ is a weak solution of $(\mathrm{IV}; x, y)$ if and only if for every $v \in D(A^*)$ the function $\langle w(t), v\rangle$ is differentiable on $[0, T]$, $(d/dt)\langle w(t), v\rangle$ is absolutely continuous on $[0, T]$ and

$$(1) \qquad \begin{cases} (d^2/dt^2)\langle w(t), v\rangle=\langle w(t), A^*v\rangle+\langle g(t), v\rangle & \text{a.e.} t \in [0, T], \\ w(0)=x \quad \text{and} \quad (d/dt)\langle w(t), v\rangle|_{t=0}=\langle y, v\rangle. \end{cases}$$

Our theorem is now stated as follows:

Theorem. *There exists for each pair* $[x, y] \in X \times X$ *a unique weak solution* $w(t)$ *of* $(\mathrm{IV}; x, y)$ *if and only if* A *is the infinitesimal generator of a cosine family* $\{C(t); t \in R=(-\infty, \infty)\}$ *on* X, *and in this case* $w(t)$ *is given by*

$$(2) \qquad w(t)=C(t)x+S(t)y+\int_0^t S(t-s)g(s)ds, \qquad t \in [0, T],$$

where $\{S(t); t \in R\}$ *is the sine family associated with* $\{C(t); t \in R\}$.

Remark. Let $B(X)$ denote the set of all bounded linear operators from X into itself. A one-parameter family $\{C(t); t \in R\}$ in $B(X)$ is called a *cosine family* if it satisfies the following conditions:

(i) $C(s+t)+C(s-t)=2C(s)C(t)$ for all $s, t \in R$;

(ii) $C(0)=I$ (the identity operator);

(iii) $C(t)x: R \to X$ is continuous for every $x \in X$.