27. On Partial Isometries

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1. Introduction. We shall show the relation among paranormality, quasinormality and property of partial isometry and we shall give some necessary conditions on operators which are similar to partial isometries and finally we shall give certain sufficient conditions which imply partial isometries. An operator T means a bounded linear operator on a complex Hilbert space H. The numerical range $W(T)$ is defined by: $W(T)=\{(Tx, x); \|x\|=1, x \in H\}$ and $\overline{W(T)}$ means the closure of $W(T)$. A unitary operator U is said to be *cramped* if its spectrum is contained in some semicircle of the unit circle, that is,

$$\sigma(U) \subset \{e^{i\theta}; \theta_1 \leqq \theta \leqq \theta_2, \theta_2-\theta_1<\pi\},$$

where $\sigma(T)$ stands for the spectrum of T. An operator T is said to be *partial isometry* if $T=TT^*T$, *quasinormal* [6] if $TT^*T=T^*TT$, *subnormal* if T has a normal extension and *quasihyponormal* if $\|TTx\| \geqq \|T^*Tx\|$. An operator T is said to be *paranormal* [1], [3], [5], [7] if $\|T^2x\|\cdot\|x\| \geqq \|Tx\|^2$ and *k-paranormal* if $\|x\|^{k-1}\cdot\|T^kx\| \geqq \|Tx\|^k$ [5].

2. Statement of results. Theorem 1. *If T is a partial isometry, then the following six conditions are equivalent*:

(1) *T is quasinormal,*
(2) *T is subnormal,*
(3) *T is hyponormal,*
(4) *T is quasihyponormal,*
(5) *T is paranormal,*
(6) *T is k-paranormal.*

Theorem 1 is an exact precision of Halmos since the equivalent relation between (2) and (3) is cited in the proof of [6, Problem 161].

Corollary 1. *If T is a partial isometry such that T^* is k-paranormal, then T is the direct sum of a co-isometry and zero.*

Corollary 1 is a generalization of the following "*if T is an isometry such that T^* is paranormal, then T is unitary*" [9, Lemma 4].

Theorem 2. *If T is similar to a partial isometry V with the initial projection P and if the similarity could be implemented by an operator A commuting with P, then there exists S satisfying $0 \notin \overline{W(S)}$, $SP=PS$ and the following $T^*ST=SP$ hold.*

Corollary 2. *Let T be an operator satisfying the hypothesis of Theorem 2. Then there exist T_1 and S satisfying $0 \notin \overline{W(S)}$, $SP=PS$*