26. On the Bellman Transform of the Coefficients of Some Special Sine-series

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§1. Let $\{c_n\}$ be an infinite sequence of real numbers, and let $(Tc)_n$ denote the n-th arithmetic mean of $\{c_n\}$, i.e.

$$(Tc)_n=\frac{1}{n}\sum_{k=1}^{n} c_k.$$

It was Hardy [5] who proved that if

$$\sum_{n=1}^{\infty} c_n \sin nx \tag{1}$$

is the Fourier series of some L^p-function $f(x)\in L^p$, $p\geqq 1$, then

$$\sum_{n=1}^{\infty} (Tc)_n \sin nx \tag{2}$$

is the Fourier series of some L^p-function.

Bellman [2] introduced the transform

$$(T^*c)_n=\sum_{k=n}^{\infty}\frac{c_k}{k},$$

and proved that if (1) is the Fourier series of an $f(x)\in L^p$, $p>1$, then

$$\sum_{n=1}^{\infty} (T^*c)_n \sin nx \tag{3}$$

is the Fourier series of the class L^p. We note that we cannot here put $p=1$ in general, as is easily seen from the example

$$\sum_{n=1}^{\infty}\frac{\sin nx}{\log^2 (n+1)}.$$

It seems still open to find the necessary and sufficient condition for (3) being the Fourier series of an L^1-function when (1) is the Fourier series of an $f(x)\in L^1$. The object of this note is to provide such necessary and sufficient conditions in the special case when $\{c_n\}$ is of bounded variation,[1] i.e.

$$\sum_{n=1}^{\infty} |\Delta c_n|<\infty, \tag{4}$$

where $\Delta c_n=c_n-c_{n+1}$.

We remark that for this special sequence $\{c_n\}$ G. and S. Goes [4] proved that a necessary and sufficient condition for (2) being the Fourier series of an L^1-function is

$$\sum_{n=1}^{\infty}\frac{|c_n|}{n}<\infty. \tag{5}$$

1) An infinite sequence of bounded variation converges to a finite limit.