14. *On Commutative Rings which have Completely Reducible Torsion Theories*

By Kiyoichi OSHIRO
Department of Mathematics, Yamaguchi University

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Throughout this note we assume that R is a commutative ring with identity and all R-modules are unital. Let (T, F) be a torsion theory for R-mod. We call that $T(F)$ is completely reducible if every R-module belonging to $T(F)$ is completely reducible, and call (T, F) completely reducible when both T and F are completely reducible.

The purpose of this paper is to study completely reducible torsion theories. It is shown in Theorem 2.5 that if R has a completely reducible torsion theory, then it is a (von Neumann) regular ring whose spectrum of prime ideals has only a finite number of non-isolated points and in this case all completely reducible torsion theories are determined by a finite subset of the spectrum containing all non-isolated points.

A torsion theory is a pair (T, F) of subclasses of R-mod satisfying

(1) $T \cap F = \{0\}$,

(2) T is closed under homomorphic images,

(3) F is closed under submodules, and

(4) for each A in R-mod, there is a submodule $T(A)$ of A called the torsion submodule of A such that $T(A) \in T$ and $A/T(A) \in F$.

T is then called the torsion class and F is called the torsion-free class.

A torsion theory (T, F) is called hereditary when T is closed under submodules.

Let (T, F) be a torsion theory. $T(F)$ is said to be a TTF-class if there is a subclass $U \subseteq R$-mod for which (U, T) $((F, U))$ forms a torsion theory. F is a TTF-class iff F is closed under homomorphic images.

1. We denote the Boolean ring consisting of all idempotents in R by $B(R)$ and the spectrum of prime ideals of $B(R)$ by $X(R)$. $X(R)$ forms a Boolean space with the family $\{U(e) \mid e \in B(R)\}$ as an open basis, where $U(e) = \{x \in X(R) \mid e \in x\}$. For an R-module A and x in $X(R)$, we set $Ax = \{ae \mid a \in A,\ e \in x\}$. Remark that all factor rings R/Rx for x in $X(R)$ are indecomposable as a ring, and that x in $X(R)$ is an isolated point iff $x = B(R)(1-e)$ for some minimal idempotent e in R.

We need the following lemmas for the later use, but we omit the proofs.