82. Elementary Proof of the Unique Factorization Theorem in Regular Local Rings

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As is well known, the proof of the unique factorization theorem in regular local rings of dimension d is trivial for $d=1,2$. The case $d\geq 4$ was reduced to the case $d=3$ by Zariski-Nagata [2], and the case $d=3$ was proved by Auslander-Buchsbaum [1]. The proofs in [1], [2] depend on homological method. The author gave an ideal-theoretic proof of the result of [2] in [3]. The purpose of the present paper is to show that also the result of [1] can be proved in an elementary way, without referring to any general theory of homological algebra, along the same idea as in [3].*)

For the convenience of proof, we shall state here the following well-known propositions without any proof.

Proposition 1. Let F be a finite free module over a Noetherian ring, then every submodule of F has a finite base.

Proposition 2. Let M be a finite module over a local ring Q. Let M_0 be a submodule of M, and $\mathfrak{a}$ a proper ideal of Q. If $M\subseteq M_0+\mathfrak{a}M$, then $M=M_0$.

We first prove the following lemmas.

Lemma 1. Let $\mathfrak{q}$ be a primary ideal belonging to the maximal ideal $\mathfrak{m}=Qu+Qv$ of a regular local ring Q of dimension 2. If $\mathfrak{q}$ includes u, then there exists an element b of $\mathfrak{q}$ such that $\mathfrak{q}=Qb+Qu$.

Proof. Since the residue ring $\overline{Q}=Q/Qu$ is a one-dimensional regular local ring, it follows that $\bar{\mathfrak{q}}=\mathfrak{q}/Qu$ is a principal ideal of $\overline{Q}$, whence follows the conclusion.

Lemma 2. Let $\mathfrak{q}$ be a primary ideal belonging to the maximal ideal $\mathfrak{m}$ of a regular local ring Q of dimension 2, and let $\{a_1,a_2,\cdots,a_n\}$ be its minimal base. Let $X_1,X_2,\cdots,X_n$ be indeterminates, and $F=QX_1+QX_2+\cdots+QX_n$ a free module over Q. Let $0\rightarrow R\rightarrow F\xrightarrow[\varphi]{}\mathfrak{q}\rightarrow 0$ be an exact sequence, where φ induces the mapping $\varphi(X_i)=a_i;\ i=1,2,\cdots,n$. Then R is a free module over Q.

Proof. It is evident that there exists an element u of a minimal base of $\mathfrak{m}$ such that $a_1,a_2,\cdots,a_n\notin Qu$. Let $\mathfrak{a}_1=Qa_2+Qa_3+\cdots+Qa_n$, $\mathfrak{a}_2=Qa_3+Qa_4+\cdots+Qa_n$, $\cdots$, $\mathfrak{a}_{n-2}=Qa_{n-1}+Qa_n$, $\mathfrak{a}_{n-1}=Qa_n$, then $\mathfrak{a}_1+\mathfrak{q}u$,

*) Recently Nagata proved syzygy theory of local rings without using homological algebra. His book including the theory is in press.