79. Three Primes in Arithmetical Progression

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1. One of the long-standing conjectures on the distribution of prime numbers states that there are infinitely many m-plets of primes $p_1, p_2, \cdots, p_m$ in arithmetical progression for every $m>2$, which is, at least, empirically true. Unfortunately, however, we cannot at present prove (or disprove) the existence of such an m-plet of primes for an unspecified value of the number m. Some sequences of prime numbers are known to be in arithmetical progression. For example, the ten numbers

$$119+210\,n \qquad (n=0, 1, 2, \cdots, 9)$$

are all primes (cf. [1]).

Our aim in the present note is to show that there exist infinitely many triplets of primes p_1, p_2, p_3 in arithmetical progression, i.e. such that $p_1<p_3$ and

$$p_1+p_3=2p_2.$$

In fact, we can prove somewhat more. Let a be a positive integer, b an arbitrary integer, and let $N(x, a, b)$ denote the number of solutions of

$$p_1+p_3=ap_2+b$$

in prime numbers p_1, p_2, p_3 with $2\leqq p_j\leqq x$ $(j=1, 2, 3)$. Then there holds the following

Theorem. *We have*

$$N(x, a, b)=C(a, b)\ T(x, a, b)+O(x^2(\log\ x)^{-A}) \quad (x\to\infty)$$

for every $A>3$, where the O-constant depends possibly on a, b and A and where

$$C(a, b)=\prod_{p\mid a,\, p\mid b}\frac{p}{p-1}\prod_{\substack{p\mid a,\, p\nmid b\\ \text{or}\\ p\nmid a,\, p\mid b}}\frac{p(p-2)}{(p-1)^2}\prod_{p\nmid ab}\left(1+\frac{1}{(p-1)^3}\right);$$

$$T(x, a, b)=\sum(\log\ n_1\ \log\ n_2\ \log\ n_3)^{-1},$$

the summation being extended over all integer solutions n_1, n_2, n_3 of the equation

$$n_1+n_3=an_2+b$$

with $2\leqq n_j\leqq x$ $(j=1, 2, 3)$.

It is easy to see from our result that $C(a, b)>0$ unless a and b have a different parity and, in particular, we have

$$C(2, 0)\geqq 2(\zeta(2))^{-1}=\frac{12}{\pi^2}.$$

Also, it should be noted that