On the Inequality of Ingham and Jessen.

By Hiraku TÔYAMA.

Mathematical Institute, Tokyo Institute of Technology.

(Comm. by T. KUBOTA, M.J.A., Nov. 12, 1948.)

Minkowski's inequality was formulated by Ingham and Jessen in the following symmetrical form.[1]

Let A be a m-rowed and n-columned matrix with non-negative elements

$$A=\begin{pmatrix} a_{11} & a_{12} \cdots\cdots a_{1n} \\ a_{21} & a_{22} \cdots\cdots a_{2n} \\ \vdots & \vdots \qquad \vdots \\ a_{m1} & a_{m2} \cdots\cdots a_{mn} \end{pmatrix}$$

Then the following inequality holds:

$$\Big(\sum_{\mu=1}^{m}\Big(\sum_{\nu=1}^{n}a_{\mu\nu}^{r}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}} \leqq \Big(\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{m}a_{\mu\nu}^{s}\Big)^{\frac{r}{s}}\Big)^{\frac{1}{r}},$$

if $0<r<s<\infty$.

We write this inequality in the form of quotient

$$1\leqq\frac{\Big(\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{m}a_{\mu\nu}^{s}\Big)^{\frac{r}{s}}\Big)^{\frac{1}{r}}}{\Big(\sum_{\mu=1}^{m}\Big(\sum_{\nu=1}^{n}a_{\mu\nu}^{r}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}}},$$

and wish to evaluate this quotient form on the right-hand side. The result is the

Theorem

$$\frac{\Big(\sum_{\nu=1}^{n}\Big(\sum_{\mu=1}^{m}a_{\mu\nu}^{s}\Big)^{\frac{r}{s}}\Big)^{\frac{1}{r}}}{\Big(\sum_{\mu=1}^{m}\Big(\sum_{\nu=1}^{n}a_{\mu\nu}^{r}\Big)^{\frac{s}{r}}\Big)^{\frac{1}{s}}}\leqq \text{Min.}\,(m, n)^{\frac{1}{r}-\frac{1}{s}}.$$

The constant on the right side is the best possible.

At first we suppose $r=1$, $s>1$ and prove a simple lemma, which can be easily verified with the help of elementary calculus.

Lemma. Let $0\leqq x\leqq c$, $0\leqq y\leqq c$ be variables, whose sum is constant: $x+y=c$, then the function

$$f(x, y)=(x^s+a^s)^{\frac{1}{s}}+(y^s+b^s)^{\frac{1}{s}}$$

attains its maximum only at the extremity of the interval $(0, c)$.

Proof.

$$f(x, y)=f(x, c-x)=g(x)$$

1) Hardy-Littlewood-Polya, Inequalities, p. 31