67. A Note on Extensions of Groups.

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1. If a group $\mathfrak{G}$ contains a normal subgroup $\mathfrak{N}$ and $\mathfrak{G}/\mathfrak{N}$ is isomorphic to $\mathfrak{A}$, we call $\mathfrak{G}$ an *extension of* $\mathfrak{N}$ *by* $\mathfrak{A}$. The problem of extension is to obtain all extensions of $\mathfrak{N}$ by $\mathfrak{A}$ when $\mathfrak{N}$ and $\mathfrak{A}$ are given. The conditions to determine every extension were at first given by O. Schreier[1] and afterwards by K. Shoda[2] in another way.

This note is devided in two parts. In section 2, we shall show that the problem of extension can be reduced in a sense to the case when $\mathfrak{N}$ is abelian, and in section 3, we shall consider central extensions of $\mathfrak{N}$ by $\mathfrak{A}$ under the assumption that $\mathfrak{N}$ and $\mathfrak{A}$ are both abelian, where an extension of $\mathfrak{N}$ by $\mathfrak{A}$ is called a ***central extension*** when $\mathfrak{N}$ is contained in its center.

2. From the theorem of O. Schreier, any extension of $\mathfrak{N}$ by $\mathfrak{A}$ may be determined by a factor set $\{C_{a,b}\}$ and a homomorphic mapping $\bar{\sigma}$ of $\mathfrak{A}$ into the residue class group of the automorphism group of $\mathfrak{N}$ by its inner automorphism group. We shall call such extension a ***$\bar{\sigma}$-extension.*** Let σ be a mapping from $\mathfrak{A}$ into the automorphism group of $\mathfrak{N}$ such that the residue class containing σ a) is equal to $\bar{\sigma}$ a). Then any $\bar{\sigma}$-extension may be determined by a factor set $\{C_{a,b}\}$ which satisfies the following conditions:

1) $A^{\sigma(a)\sigma(b)}=C_{a,b}^{-1}A^{\sigma(ab)}C_{a,b}$ $(A \in \mathfrak{N};\ a,b \in \mathfrak{A})$

2) $C_{ab,c}C_{a,b}^{\sigma(c)}=C_{a,bc}C_{b,c}$.

We shall call such factor set a *σ-factor set.*

Theorem 1. *Let $\{C_{a,b}\}$ and $\{D_{a,b}\}$ be two σ-factor sets. Then the set $\{Z_{a,b}=D_{a,b}C_{a,b}^{-1}\}$ is contained in the center $\mathfrak{Z}$ of $\mathfrak{N}$ and satisfies the following conditions:*

3) $Z_{ab,c}Z_{a,b}^{\sigma(c)}=Z_{a,bc}Z_{b,c}$.

Conversely, if $\{C_{a,b}\}$ is a σ-factor set, and if $\{Z_{a,b}\}$ is contained in $\mathfrak{Z}$ and satisfies 3), then $\{D_{a,b}=C_{a,b}Z_{a,b}\}$ is a σ-factor set.

Proof. If $\{C_{a,b}\}$ and $\{D_{a,b}\}$ are both σ-factor sets, then from 1) $C_{a,b}^{-1}AC_{a,b}=D_{a,b}^{-1}AD_{a,b}$ for any $A \in \mathfrak{N}$, hence $Z_{a,b}=D_{a,b}C_{a,b}^{-1} \in \mathfrak{Z}$. Further, since

1) O. Schreier: Über die Erweiterung von Gruppen, Monatshefte für Math. u. Phisik, 34 (1926) 321-346.

2) K. Shoda: Über die Schreiersche Erweiterungstheorie. Proc. Acad. Tokyo (1943) 518-519.