35. On Integral Representations of Bilinear Functionals.

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Let E_1 and E_2 be Banach spaces. Let $A = A(f, g)$ $(f \in E_1, g \in E_2)$ be a functional defined in (E_1, E_2) such that

1. $A(f, g)$ is additive and homogeneous concerning f and g,
2. $A(f, g)$ is bounded; that is, there is a constant M such that

$$|A(f, g)| \leqq M \|f\| \cdot \|g\| .$$

for all $f \in E_1$, and all $g \in E_2$.

Such functional $A(f, g)$ of f and g is called *bilinear* in (E_1, E_2), and the greatest lower bound of M is called the *norm* of the functional A and is denoted by $\|A\|$. It is then evident that

$$\|A\| = \sup_{\|f\|=1,\ \|g\|=1} |A(f, g)| .$$

Integral representations of bilinear functionals have been studied by Prof. Izumi [1], but their norms were not exactly given. In this note we will give the exact form of their norms. For this purpose we first prove that the representation problem of bilinear functionals in (E_1, E_2) is equivalent to that of the linear operations from E_1 to $\bar{E}_2$ (or from E_2 to $\bar{E}_1$), where $\bar{E}$ denotes the conjugate space of E. Therefore we get the representations of linear operations between some concrete Banach spaces from results of Prof. Izumi [1]. On the other hand, we obtain the general form of bilinear functionals from the known representations of the linear operations.

Theorem 1. *The general form of the bilinear functional A in (E_1, E_2) is derived from the representation of the linear operation U from E_1 to $\bar{E}_2$ (or from E_2 to $\bar{E}_1$) and vice versa. The norm $\|A\|$ equals to the norm $\|U\|$.*

Proof. Let $A(f, g)$ be a bilinear functional, where $f \in E_1$, $g \in E_2$, then $|A(f, g)| \leqq \|A\| \cdot \|f\| \cdot \|g\|$. Therefore, if f is fixed, $|A(f, g)| \leqq M \|g\|$, that is $A(f, \cdot) \in \bar{E}_2$. Since $A(f, \cdot)$ is additive concerning f and

$$(1) \qquad \|A(f, \cdot)\| = \sup_{\|g\|=1} |A(f, g)| \leqq \|A\| \cdot \|f\| ,$$

$Uf \equiv A(f, \cdot)$ is the linear operation from E_1 to $\bar{E}_2$. Similarly $U'g \equiv A(\cdot, g)$ is the linear operation from E_2 to $\bar{E}_1$. Thus to any bilinear functional A in (E_1, E_2) corresponds a linear operation U from E_1 to $\bar{E}_2$ (or from E_2 to $\bar{E}_1$), and from (1)