123. *Simple Proof of a Theorem of Ankeny on Dirichlet Series.*

By Koziro IWASAKI.
University of Tokyo, Tokyo.
(Comm. by Z. SUETUNA, M.J.A., Dec. 12, 1952.)

Mr. Ankeny has proved recently the following theorem[1]:

Let $\chi_1, \chi_2, \cdots, \chi_n$ be n primitive Dirichlet characters with at most one of them being principal, and let $L(s; \chi_i)$ be Dirichlet L-series corresponding to χ_i. If then all coefficients of the Dirichlet series

$$Z(s)=\prod_{i=1}^{n} L(s; \chi_i)$$

are non-negative, then $Z(s)$ is the Dedekind ζ-function of some Abelian extension of the rational number field.

In the following I shall give a simple proof of this interesting theorem. Let H be the set of all distinct characters among $\chi_1\ \chi_2, \cdots, \chi_n$, and G the group of characters generated by H. Further let a_χ be the number of times χ, one of the elements of G, appears in $\{\chi_1, \chi_2, \cdots, \chi_n\}$ and f the least common multiple of all the conducters of our characters. It is easily shown that

$$\begin{aligned} Z(\mathrm{s}) &= \prod_{\chi\in H} L(s; \chi)^{a_\chi} \\ &= \exp\Big(\sum_{p}\sum_{g=1}^{\infty}\big(\sum_{\chi\in H} a_\chi \chi(p^g)/gp^{gs}\big)\Big). \end{aligned}$$

So we get as the coefficient of $1/p^s$ in $Z(s)$

$$\sum_{\chi\in H} a_\chi\chi(p)=\sum_{\chi\in G} a_\chi\chi(p).$$

By the hypothesis of our theorem

$$\sum_{\chi\in G} a_\chi\chi(p)\geqq 0 \qquad \text{for all prime numbers } p,$$

so that we have by Dirichlet's prime number theorem

$$F(u)=\sum_{\chi\in G} a_\chi\chi(u)\geqq 0 \tag{1}$$

for each u of the representatives of the reduced classes mod f.

From (1) we get

$$ga_\chi=\sum_{u \bmod f} \chi^{-1}(u)F(u), \tag{2}$$

where g is the order of G; in particular

$$ga_{\chi_0}=\sum_{u \bmod f} F(u), \qquad (u, f)=1,$$