248. *Representations of 1-homology Classes of Bounded Surfaces*

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Let $F_{p,q}$ be a compact connected orientable surface of genus p with q boundary components, where q may be equal to 0. Let $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_p, \boldsymbol{c}_1, \cdots, \boldsymbol{c}_{q-1}$ be standard generators for the 1-dim. integral homology group $H_1(F_{p,q}; Z)$. Here the $\boldsymbol{c}_i$ correspond to consistently oriented boundary circles (one is omitted because it is homologous to the sum of the others), and the $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are standard curves on $F_{p,q}$, chosen so that $\boldsymbol{a}_i \cap \boldsymbol{a}_j = \boldsymbol{b}_i \cap \boldsymbol{b}_j = \boldsymbol{a}_i \cap \boldsymbol{b}_j = \emptyset$ if $i \neq j$ and $\boldsymbol{a}_i, \boldsymbol{b}_i$ intersect nicely at one point.

In the case $q=0$, T. Kaneko [1], [2] proved the following theorem:

Kaneko's Theorem. *A non-zero homology class* $\sum_{i=1}^{p} \alpha_i \boldsymbol{a}_i + \sum_{i=1}^{p} \beta_i \boldsymbol{b}_i$ *of* $H_1(F_{p,0}; Z)$ *is representable by a simple closed curve on* $F_{p,0}$ *if and only if the greatest common divisor* $(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_p) = 1$.

In the note, we generalize this theorem for bounded case as follows:

Theorem. *A non-zero homology class* $\sum_{i=1}^{p} \alpha_i \boldsymbol{a}_i + \sum_{i=1}^{p} \beta_i \boldsymbol{b}_i + \sum_{i=1}^{q-1} \gamma_i \boldsymbol{c}_i$ *of* $H_1(F_{p,q}; Z)$ *is representable by a simple closed curve on* $F_{p,q}$ *if and only if one of the following two conditions is satisfied*:

(1) *Not all the* α_i *and* β_i *are zero and the g.c.d.* $(\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_p) = 1$,

(2) $\alpha_i = \beta_i = 0$ *for* $1 \leqq i \leqq p$ *and* $|\gamma_i| \leqq 1$ *for* $1 \leqq i \leqq q-1$ *and all non-zero* γ_i *have the same sign.*

Proof of necessity. Let l be an oriented simple closed curve in $\mathcal{J}F_{p,q}$.[1] Fill in each boundary component with a disk, obtaining $F_{p,0}$.

If now l is not homologous to zero on $F_{p,0}$, by Kaneko's Theorem we fall into case (1) since $\boldsymbol{a}_1, \cdots, \boldsymbol{a}_p, \boldsymbol{b}_1, \cdots, \boldsymbol{b}_p$ form standard generators for $H_1(F_{p,0}; Z)$.

If l is now homologous to zero on $F_{p,0}$, l bounds a surface $F_{p',1}$ on $F_{p,0}$ where $0 \leqq p' \leqq p$, and so l separates $F_{p,q}$ into two surfaces, say $F_{p',q'}$ and $F_{p-p',q+2-q'}$. On $F_{p',q'}$, l or $-l$ is homologous to the sum of $\partial F_{p',q'} - l \subset \partial F_{p,q}$, so we fall into case (2) above.

Proof of sufficiency. We shall need the following elementary lemma.

Lemma. *A homology class* $\boldsymbol{a}_1 + \gamma \boldsymbol{c}_1$ *is representable by a simple*

1) $\mathcal{J}$=interior, ∂=boundary.