146. On a Mean Value Theorem for the Remainder Term in the Prime Number Theorem for Short Arithmetic Progressions

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1. In 1965 Bombieri [1] improved almost ultimately the large sieve of Linnik and Rényi, and as an application he derived an astounding result on the average size of the remainder term in the prime number theorem for arithmetic progressions. Recently Jutila [4] has proved an analogous result for short arithmetic progressions, i.e. he considered the estimation of the expression

$$D(Q\,;x,h)=\sum_{q\le Q}\max_{(l,q)=1}\left|\psi(x+h\,;q,l)-\psi(x\,;q,l)-\frac{h}{\varphi(q)}\right|, \tag{1.1}$$

where $\psi(x\,;q,l)$ is the usual Čebyšev function for the arithmetic progression $\equiv l \bmod q$ and Q, h are appropriate functions of x.

Both results of Bombieri and Jutila have been obtained by reducing the problem to the estimation of the total density of zeros of 'many' L-functions. But Gallagher [2] has found a way to prove Bombieri's result without using the density theorem. In [4] an opinion is expressed that it seems difficult to prove a non-trivial estimation of (1.1) on the similar line. The purpose of the present paper is to offer such a proof.

Our main tool is the following beautiful inequality of Gallagher [3]: If

$$\sum_n n|a_n|^2<+\infty,$$

then we have

$$\sum_{q\le Q}\sum_{\chi \bmod q}{}^{*}\int_{-T}^{T}\left|\sum_n a_n\chi(n)n^{-it}\right|^2 dt \ll \sum_n (Q^2T+n)|a_n|^2, \tag{1.2}$$

where $\sum^*$ denotes a sum over all primitive characters mod q.

2. Before entering into the proof we list up here some definitions. Let $\Lambda(n)$ be the von Mangold function and let

$$\psi(x,\chi)=\sum_{n\le x}\chi(n)\Lambda(n).$$

Let $\mu(n)$ be the Möbius function and let

$$H(s,\chi)=\sum_{n\le Q^2T}\mu(n)\chi(n)n^{-s}.$$

Let M be the number defined by