186. *A Semigroup-Theoretic View of Projective Class Groups*

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This note has arisen from an interest in subsumming the basic description of the projective class groups of rings and their representation as quotients of Grothendieck groups into the elementary theory of commutative semigroups. This goal is reached in § 2 by means of a method given in § 1 of constructing quotient monoids modulo subtractive subsemigroups. The quotient construction of § 1 provides a convenient vocabulary for a discussion of greatest monoid images which is given in § 3. An interpretation of § 1 and § 3 in terms of category concepts is appended as § 4. All semigroups considered here will be commutative and additive notation will be used.

1. The quotient monoids. A subsemigroup B of a commutative semigroup A is *subtractive* if whenever $a+b=b'$, for $a \in A$ and b, $b' \in B$, we have $a \in B$. To give familiarity with the definition we list three elementary observations: The subtractive subsemigroups of a group are precisely the subgroups. A proper ideal is never subtractive. If C is a subtractive subsemigroup of B and B is a subtractive subsemigroup of A, then C is subtractive in A.

Let B be a subtractive subsemigroup of a commutative semigroup A. We use B to define a relation $\rho(B)$ in A: For each a, $a' \in A$ we write $a\rho(B)a'$ if there are b, $b' \in B$ for which $a+b=a'+b'$. It is easy to verify that $\rho(B)$ is a congruence relation and that whenever we have $a\rho(B)b$, for $a \in A$ and $b \in B$, we have $a \in B$. We denote the quotient semigroup $A/\rho(B)$ by the shorter form A/B and we observe that A/B is a monoid with B as identity. If A is a group, then A/B coincides with its usual meaning in group theory.

For each homomorphism $h: A \to M$ of a commutative semigroup A onto a monoid M, we define the kernel of h to be the subset $\ker(h) = \{a \in A \mid h(a)=0\}$. We list three elementary observations concerning kernels: Each kernel is a subtractive subsemigroup. For each subtractive subsemigroup B of each commutative semigroup A, the kernel of the natural map $A \to A/B$ is B. A subsemigroup B of a commutative semigroup A is the kernel of a homomorphism of A onto a monoid if

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