129. One Condition for R(K)=A(K)

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We will show here one condition to coincide $A(K)$, all continuous functions on the compact plane set K which are analytic in $\mathring{K}$, and $R(K)$, all those functions on K which are approximable by rational functions with poles off K. This sharpens the result of Theorem 4.1 in [3].

Let U be a bounded open set in the complex plane C, $\overline{U}$ be the closure of U, and ∂U be the boundary of U. Let $A(U)$ be all continuous functions on $\overline{U}$ which are analytic in U and $R(U)$ be all those functions which are approximable uniformly on $\overline{U}$ by rational functions with poles off $\overline{U}$. Let $H^\infty(U)$ be the uniform algebra of all bounded analytic functions on U.

Lemma 1. *Let B be a subalgebra in $H^\infty(U)$ which contains $A(U)$. Then there is a continuous map from the maximal ideal space M_B of B onto $\overline{U}$.*

Proof. The coordinate function Z belongs to B and the Gelfand transform $\hat{Z}$ of Z is the desired map. For since $B \subseteq H^\infty(U)$, every homomorphism in the maximal ideal space of $H^\infty(U)$ determines a homomorphism in M_B by restricting it to B. So $\hat{Z}(M_B)$ contains $\overline{U}$. Suppose $\lambda \notin \overline{U}$, then $(z-\lambda)^{-1} \in A(U)$, that is, $z-\lambda$ is invertible in B. Thus $\varphi(z-\lambda) \neq 0$ for all $\varphi \in M_B$. Hence λ does not belong to $\hat{Z}(M_B)$ and $\hat{Z}(M_B) = \overline{U}$. This completes the lemma.

The analogous result is valid by replacing the algebra $A(U)$ by the algebra $R(U)$.

For B as above, we denote the fibers $M_\lambda(B)$ of M_B over points $\lambda \in \overline{U}$ by

$$M_\lambda(B) = \{\varphi \in M_B\,;\, \varphi(z) = \lambda\}.$$

If $\lambda \in U$, then $M_\lambda(B)$ consists of the single homomorphism.

Lemma 2. *Let B be as above lemma. Then for each $\lambda \in \partial U$ and for each $f \in A(U), \varphi(f) = f(\lambda)$ for all $\varphi \in M_\lambda(B)$.*

Proof. As seen in [2], by using the Vitushkin's operator, we can find a bounded sequence $f_n \in A(U)$ which is analytic at $\{\lambda\}$ and the f_n converges uniformly to f on $\overline{U}$. So it is sufficient to show the case that $g \in A(U)$ is analytic at $\{\lambda\}$. If $g \in A(U)$ is analytic at $\{\lambda\}$, then

$$\frac{g(z)-g(\lambda)}{z-\lambda} \in A(U). \quad \text{Hence } \frac{g(z)-g(\lambda)}{z-\lambda} \in B. \quad \text{Thus } \varphi(g) = g(\lambda)$$

for all $\varphi \in M_B$ and $\varphi(z) = \lambda$. And the lemma is proved.