18. Conjugate Spaces of Operator Algebras

By Zirô TAKEDA
Mathematical Institute, Tôhoku University
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By the representation theorem for (AL)-space,[4] the conjugate space $R(\Omega)$ of the Banach space $C(\Omega)$ composed of all continuous complex valued functions vanishing at infinity on a locally compact Hausdorff space Ω i.e. the space of all bounded complex Radon measures on Ω is isomorphic to $L^1(\Gamma)$ on a suitable localizable measure space Γ. Hence the conjugate space of $R(\Omega)$ is isomorphic to $L^\infty(\Gamma)$. As the measure space Γ is localizable, $L^\infty(\Gamma)$ considered as the set of multiplication operators on $L^2(\Gamma)$ is a maximal abelian subring in the ring of all bounded operators on $L^2(\Gamma)$,[8] this implies $L^\infty(\Gamma)$ is a weakly closed operator algebra i.e. a W^*-algebra. On the other hand, the double conjugate space of the Banach space composed of all completely continuous operators on a Hilbert space H is isomorphic to the space $B(H)$ of all bounded operators on H.[2][7] From these special cases, we get naturally the following conjecture: *Is the double conjugate space of a uniformly closed self-adjoint operator algebra or equivalently a B^*-algebra always isomorphic to a W^*-algebra considered as a Banach space?* The affirmative of this conjecture was announced by S. Sherman,[9] but its detailed proof is not published yet now. In § 2 of this note, we give a proof of this theorem. By a letter from S. Sherman the author learned that his original proof is essentially same with our own. In this occasion, we want to express our hearty thanks for his kind regards. Recently J. Dixmier has shown that a W^*-algebra considered as a Banach space is always isomorphic to the conjugate space of all ultra-weakly continuous linear functionals.[3] Observing this remarkable property of W^*-algebras, we give in § 3 a characterization of W^*-algebras. Even though this characterization does not depend on the algebraic structure of the algebra,[5] it seems for us to have some interestings especially from the view point of the non-commutative integration theory. The detailed explanations of this point will be shown in the forthcoming paper.

1. In this section, we give a theorem due to J. Dixmier as a preparation for the followings. Let $A, \bar{A}$ be a Banach space and its conjugate space respectively and by $\sum, \bar{\sum}$ denote the unit sphere of each space. For a closed subspace V of $\bar{A}$ we define its *characteristic* r by