59. A Generalization of Ascoli's Theorem

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Let R be an abstract space. For a double system of mappings $\mathfrak{a}_{\gamma,\lambda}$ of R into uniform spaces S_λ $(\gamma \in \Gamma_\lambda, \lambda \in \Lambda)$, there exists the weakest uniformity $\mathfrak{U}$ on R for which $\mathfrak{a}_{\gamma,\lambda}(\gamma \in \Gamma_\lambda)$ is equi-continuous for every $\lambda \in \Lambda$. In an earlier paper[1] we have obtained a condition for which R is complete by $\mathfrak{U}$. In this paper we shall consider conditions for which R is totally bounded by $\mathfrak{U}$ and as a generalization of Ascoli's theorem, we shall prove Theorem II which is essentially more general than that obtained by N. Bourbaki.[2]

Lemma 1. *Let* $\mathfrak{a}_\nu$ $(\nu=1, 2,\ldots, n)$ *be a finite number of mappings of* R *into uniform spaces* S_ν *with uniformities* $\mathfrak{V}_\nu$ $(\nu=1, 2,\ldots, n)$ *respectively. If the image* $\mathfrak{a}_\nu(R)$ *is totally bounded in* S_ν *for every* $\nu=1, 2,\ldots, n$, *then for any* $U_\nu \in \mathfrak{V}_\nu$ $(\nu=1, 2,\ldots, n)$ *we can find a finite number of points* $a_\mu \in R$ $(\mu=1, 2,\ldots, m)$ *such that*

$$R = \sum_{\mu=1}^{m} \prod_{\nu=1}^{n} \mathfrak{a}_\nu^{-1} U_\nu(a_\mu),$$

that is, for any $x \in R$ *we can find* μ *for which*

$$\mathfrak{a}_\nu(x) \in U_\nu(\mathfrak{a}_\nu(a_\mu)) \qquad \textit{for every } \nu=1, 2,\ldots, n.$$

Proof. For any $U_\nu \in \mathfrak{V}_\nu$ $(\nu=1, 2,\ldots, n)$ we can find by definition $V_\nu \in \mathfrak{V}_\nu$ such that

$$V_\nu^{-1} \times V_\nu \leqq U_\nu \qquad (\nu=1, 2,\ldots, n).$$

Since the image $\mathfrak{a}_\nu(R)$ is totally bounded by assumption, we can find a finite number of points $y_{\nu,\mu} \in S_\nu$ $(\mu=1, 2,\ldots, m)$ such that

$$\mathfrak{a}_\nu(R) \subset \sum_{\mu=1}^{m_\nu} V_\nu(y_{\nu,\mu}) \qquad (\nu=1, 2,\ldots, n).$$

Corresponding to every system $\mu_\nu=1, 2,\ldots, m_\nu$ $(\nu=1, 2,\ldots, n)$ we select a point $a_{\mu_1\mu_2\cdots\mu_n} \in R$ such that

$$\mathfrak{a}_\nu(a_{\mu_1\mu_2\cdots\mu_n}) \in V_\nu(y_{\nu,\mu_\nu}) \qquad \text{for every } \nu=1, 2,\ldots, n,$$

if exists. Then for any $x \in R$ we can find μ_ν $(\nu=1, 2,\ldots, n)$ such that

$$\mathfrak{a}_\nu(x) \in V_\nu(y_{\nu,\mu_\nu}) \qquad \text{for every } \nu=1, 2,\ldots, n,$$

1) H. Nakano: On completeness of uniform spaces, Proc. Japan Acad., **29**, 490-494 (1953).

2) N. Bourbaki: Topologie générale, **3**, Chap. 10, espaces fonctionnels. Paris (1949).