69. On Ergodic Theorems

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1. Introduction. Let $(X, \mathfrak{B}, m)$ denote a measure space such that X is a set, $\mathfrak{B}$ is a Borel field of subsets of X, and m is a σ-finite measure defined on $\mathfrak{B}$. Let T be a single-valued, measurable and non-singular transformation of X into itself. The measurability and non-singularity of the transformation are used in the sense of Ryll-Nardzewski [3].[1] In the following, all sets in consideration are supposed $\mathfrak{B}$-measurable.

We define the following statements.

(I) *There exists a constant K such that for any set A of positive measure*

$$0<\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}A) \leqq K\cdot m(A).$$

(I′) *There exists a constant K such that for any set A*

$$\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} m(T^{-i}A) \leqq K\cdot m(A).$$

(II) *There exist a sequence of sets $\{X_j\}$ and a constant K such that*

$$X_1 \subset X_2 \subset \cdots, \quad X=\bigcup_{j=1}^{\infty} X_j, \quad m(X_j)<\infty \quad (j=1, 2, \ldots),$$

and for any set A of positive measure

$$0<\sup_j \limsup_n \frac{1}{n}\sum_{i=0}^{n-1} m(X_j \cap T^{-i}A) \leqq K\cdot m(A) \quad (j=1, 2, \ldots).$$

(II′) *There exist a sequence of sets $\{X_j\}$ and a constant K such that*

$$X_1 \subset X_2 \subset \cdots, \quad X=\bigcup_{j=1}^{\infty} X_j, \quad m(X_j)<\infty \quad (j=1, 2, \ldots),$$

and for any set A

$$\limsup_n \frac{1}{n}\sum_{i=0}^{n-1} m(X_j \cap T^{-i}A) \leqq K\cdot m(A) \quad (j=1, 2, \ldots).$$

(B) *For any function $f \in L(X, \mathfrak{B}, m)$ the limit*

$$\tilde{f}(x) = \lim_n \frac{1}{n}\sum_{i=0}^{n-1} f(T^i x)$$

exists almost everywhere (m) and $\tilde{f} \in L(X, \mathfrak{B}, m)$.

1) Numbers in square brackets refer to the references at the end of this paper.