89. *Note on an Ergodic Theorem*

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1. Let $(X, \mathfrak{B}, m)$ be a measure space such that X is a set, $\mathfrak{B}$ is a Borel field of subsets of X, and m is a σ-finite measure defined on $\mathfrak{B}$. A single valued (not necessarily one to one) transformation T of X onto itself is called *measurable* if both T and its inverse transformation T^{-1} transform every set of $\mathfrak{B}$ to a set of $\mathfrak{B}$. The measurable transformation T is called *non-singular* (with respect to m) if $E \in \mathfrak{B}$ and $m(E)=0$ imply $m(TE)=m(T^{-1}E)=0$, and is called *incompressible* (with respect to m) if $E \in \mathfrak{B}$ and $T^{-1}E \supset E$ imply $m(T^{-1}E-E)=0$. Two measures λ and μ defined on $\mathfrak{B}$ are called *equivalent* if $E \in \mathfrak{B}$ and $\lambda(E)=0$ imply $\mu(E)=0$ and conversely. A measure μ on $\mathfrak{B}$ is said to be *invariant* under the measurable transformation T (or the measurable transformation T is said to be *measure-preserving* with respect to μ) if $\mu(T^{-1}E)=\mu(E)$ for any set E of $\mathfrak{B}$.

If T is measurable and non-singular, we put $\mathfrak{B}_1=\{T^{-1}E; E \in \mathfrak{B}\}$. Then, from the Radon-Nikodym theorem, there exists a measurable function $w(x)$ such that

$$m(TE)=\int_E w(x)dm$$

for every set E of $\mathfrak{B}_1$. Let us now put

$$w_0(x)=1, \quad w_n(x)=w(x)\cdots w(T^{n-1}x)$$

for any point x of X and for $n=1, 2, \ldots$. Then we obtain the recurrence formula:

$$w_{i+j}(x)=w_i(T^j x)\cdot w_j(x)$$

for $i, j=0, 1, 2, \ldots$.

Y. N. Dowker [1][1] has offered the following question concerning the extension of Halmos' ergodic theorem [2] for one to one transformation to the case of a single valued transformation: whether, for a single valued, measurable, non-singular transformation T of X onto itself, the condition that T is incompressible (or some similar condition) yields that, for any measurable function $g(x)$ which is positive almost everywhere, the series $\sum_{i=0}^{\infty} g(T^i x) w_i(x)$ diverges almost everywhere?

1) Numbers in square brackets refer to the references at the end of this paper.