153. Remarks on the Sequence of Quasi-Conformal Mappings

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1. It seems to me that there are essentially two kinds of definition, stronger and weaker, for quasi-conformal mapping with bounded dilatation. The former is rather classical definition of Grötzsch, Teichmüller and other authors. In 1951 Pfluger suggested the latter [6], and Ahlfors remarkably improved the theory of quasi-conformal mapping by making use of it in recent few years [1–3]. The present Note, which I owe much to the investigations of Ahlfors, is concerned with relations between these definitions.

Definition 1. A topological mapping $w=f(z)$ from a domain D in the $z(=x+iy)$-plane to a domain $\varDelta$ in the $w(=u+iv)$-plane is called K-QC mapping in D, when it satisfies the following conditions there:

I) all the partial derivatives u_x, u_y, v_x, v_y exist and are continuous,

II) $$J(z)=u_xv_y-u_yv_x>0,$$

III) $$\frac{|p|+|q|}{|p|-|q|}\leq K<\infty,$$

where p, q are the complex derivatives of f

$$p(z)=f_z=\frac{1}{2}[(u_x+v_y)+i(v_x-u_y)],$$

$$q(z)=f_{\bar{z}}=\frac{1}{2}[(u_x-v_y)+i(v_x+u_y)],$$

and K is a constant ≥ 1.

Let Ω be a Jordan domain, on whose boundary four ordered points z_1, z_2, z_3, z_4, are marked in the positive sense with respect to Ω. This configuration is named *quadrilateral* and is denoted by $\Omega(z_1, z_2, z_3, z_4)$ or simply by Ω. If one maps a quadrilateral Ω by means of a sense-preserving homeomorphism $T(z)$, the image $T(\Omega)$ is again a quadrilateral. Ω can be mapped conformally onto the interior of a rectangle $0<\xi<1$, $0<\eta<\lambda$ in the $\zeta(=\xi+i\eta)$-plane, so that the points z_1, z_2, z_3, z_4 correspond to $\zeta=0, 1, 1+i\lambda, i\lambda$ respectively. By *module* of the quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ is meant the positive number λ, which shall be denoted by $\mathrm{mod}\,\Omega(z_1, z_2, z_3, z_4)$.

Definition 2. A topological mapping $w=f(z)$ which transforms a plane domain D onto another such $\varDelta$ is called a K-QC* mapping, when it satisfies the following conditions:

I′) the mapping $w=f(z)$ is sense-preserving,

II′) for any quadrilateral $\Omega(z_1, z_2, z_3, z_4)$ contained together with its boundary in D the inequality