32. A Theorem for Metrizability of a Topological Space

By Jun-iti NAGATA
Department of Mathematics, Osaka City University

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Since Alexandroff and Urysohn's work various theorems concerning metrizability of a topological space were gotten by many mathematicians, but their methods of proofs are, in general, various and rather complicated. The purpose of this brief note is to prove a theorem for metrizability, which will contain a large number of metrizability theorems as direct consequences.[1]

We use the following theorem due to E. Michael[2] as well as the well-known theorem of P. Alexandroff and P. Urysohn.

Michael's theorem. A regular topological space R is paracompact if and only if every open covering of R has an open refinement $\mathfrak{V}=\bigcup_{n=1}^{\infty}\mathfrak{V}_n$, where each $\mathfrak{V}_n$ is a locally finite collection of open subsets of R.

Theorem 1. *In order that a T_1-topological space R is metrizable it is necessary and sufficient that one can assign a nbd (=neighborhood) basis $\{U_n(x)\mid n=1,2,\cdots\}$ for every point x of R such that for every n and each point x of R there exist nbds $S_n^1(x)$, $S_n^2(x)$ of x satisfying*

i) *$y\notin U_n(x)$ implies $S_n^2(y)\frown S_n^1(x)=\phi$,*

ii) *$y\in S_n^1(x)$ implies $S_n^2(y)\subseteq U_n(x)$.*

Proof. Since the necessity is clear, we prove only the sufficiency. To begin with, R is regular, since $\overline{S_n^1(x)}\subseteq U_n(x)$. Next, to show that R is paracompact, we take an arbitrary open covering $\mathfrak{V}=\{V_\alpha\mid\alpha<\tau\}$ of R.[3] If we let

$$V_{n\alpha}=\smile\{(S_n^1(x))^\circ\mid U_n(x)\subseteq V_\alpha\},^{4)}$$
$$V_{mn\alpha}=\smile\{U_m(x)\mid U_m(x)\subseteq V_{n\alpha}\},$$
$$V'_{mn\alpha}=\smile\{S_m^1(x)\mid U_m(x)\subseteq V_{n\alpha}\},$$
$$M_{mn\alpha}=(V'_{mn\alpha}-\smile_{\beta<\alpha}V_{n\beta})^\circ\quad(m,n=1,2,\cdots,\alpha<\tau),$$

then $\mathfrak{M}_{mn}=\{M_{mn\alpha}\mid\alpha<\tau\}$ is a locally finite open collection for each m,n. To show the local finiteness of $\mathfrak{M}_{mn}$ we choose, for an arbitrary point p of R, α $(\leqq\tau)$ such that $p\in V_{mn\alpha}$, $p\notin V_{mn\beta}$ $(\beta<\alpha)$. Then it follows from the condition i) of the proposition that $S_m^2(p)\frown V'_{mn\beta}=\phi$

1) The detail of the content of this note will be published in an another place.
2) See [3].
3) In this proof we denote by τ,α,β,γ ordinal numbers.
4) A° denotes the interior of A.