30. Fourier Series. XV. Gibbs' Phenomenon

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1. Concerning Gibbs' phenomenon of the Fourier series H. Cramér [1] proved the following theorem.

Theorem 1. *There exists a number* r_0, $0<r_0<1$, *with the following property: If* $f(x)$ *is simply discontinuous at a point* ξ, *the* (C, r) *means* $\sigma_n^r(x)$ *of the Fourier series of* $f(x)$ *present Gibbs' phenomenon at* ξ *for* $r<r_0$, *but not for* $r\geqq r_0$.

On the other hand S. Izumi and M. Satô [2] proved the following theorems:

Theorem 2. *Suppose that* $f(x)=a\psi(x-\xi)+g(x)$, *where* $\psi(x)$ *is a periodic function with period* 2π *such that* $\psi(x)=(\pi-x)/2$ $(0<x<2\pi)$, *and where*

$$\limsup_{x\downarrow\xi} g(x)=0, \qquad \liminf_{x\uparrow\xi} g(x)=0,$$

$$\liminf_{x\downarrow\xi} g(x)\geqq -a\pi, \qquad \limsup_{x\uparrow\xi} g(x)\leqq a\pi,$$

$$\int_0^x |g(\xi+u)|\,du=o(|x|), \tag{1}$$

then Gibbs' phenomenon of the Fourier series of $f(x)$ *appears at* $x=\xi$.

Theorem 3. *In Theorem 2, if we replace the condition* (1) *by the following conditions*:

$$\int_0^x g(\xi+u)du=o(|x|),$$

and

$$\int_0^x \{g(t+u)-g(t-u)\}\,du=o(|x|)$$

uniformly for all t *in a neighbourhood of* ξ, *then Gibbs' phenomenon of the Fourier series of* $f(x)$ *appears at* $x=\xi$.

We proved that Theorem 1 holds even when the point ξ is the discontinuity point of the second kind, satisfying the condition in Theorem 2 [3]. More precisely,

Theorem 4. *Suppose that*

$$f(x)=a\psi(x-\xi)+g(x)$$

where $\psi(x)$ *is a periodic function with period* 2π *such that*

$$\psi(x)=(\pi-x)/2 \qquad (0<x<2\pi)$$

and where

$$\limsup_{x\downarrow\xi} g(x)=0, \qquad \liminf_{x\uparrow\xi} g(x)=0,$$

$$\liminf_{x\downarrow\xi} g(x)\geqq -a\pi, \qquad \limsup_{x\uparrow\xi} g(x)\leqq a\pi,$$