29. Fourier Series. XIV. Order of Approximation of Partial Sums and Cesàro Means

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1. It is well known that, if $f(t)$ belongs to the Lip α class, $0<\alpha<1$, then

(1) $$s_n(t, f)-f(t)=O(\log n/n^\alpha), \qquad \text{uniformly,}$$

where $s_n(t,f)$ is the nth partial sum of the Fourier series of $f(t)$.

The factor $\log n$ on the right of (1) can not be replaced by the smaller. Then arises the problem when the factor $\log n$ may be omitted. As its answer it is known the following theorems.

Theorem 1.[1] *Let $0<\alpha<1$, $p>1$, $0<\beta<1$ and $\alpha=\beta-1/p$. If $f(t)$ belongs to the Lip (β, p) class which is a subclass of Lip α, then*

(2) $$s_n(t, f)-f(t)=O(1/n^\alpha), \qquad \text{uniformly.}$$

Theorem 2.[2] *If $f(t)$ belongs to the Lip α class, $0<\alpha<1$ and it is of monotonic type, then (2) holds.*

A function $f(t)$ is said to be of monotonic type, if there is a constant C such that $f(t)+Ct$ is monotonic in the infinite interval $(-\infty, \infty)$.

We shall here treat the following problem: If $f(t)$ belongs to the Lip α class or some other classes, then under what local condition

$$s_n(x, f)-f(x)=O(1/n^\alpha)$$

holds at a point x? Similar problem arises concerning Cesàro means. The latter was recently treated by T. M. Flett [4].

The theorems which we prove are as follows.

Theorem 3. *Let $0<\alpha<1$, $p>1$. Suppose that $f(t)$ belongs to the Lip α class, or Lip (α, p) class, or that $f(t)$ is of $(1/\alpha)$-bounded variation. If the function*

$$\theta(u)=u\varphi_x(u)=u\{f(x+u)+f(x-u)-2f(x)\}$$

is of bounded variation in the right neighbourhood of $u=0$ and

(3) $$\int_0^t |d\theta(u)|=O(t^{1+\alpha}),$$

then

(4) $$s_n(x, f)-f(x)=O(1/n^\alpha).$$

This contains Theorem 2 as a particular case. We can get also a corollary of Theorem 3 which contains Theorem 2 [see § 3].

In the case $\alpha=1$ in Theorem 3, it needs an additional condition that the integral

1) Cf. [1] and [2]. Lip (α, p) is a subclass of Lip $(\alpha-1/p)$.
2) Cf. [3].