3. On a Theorem of Weyl-von Neumann

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1. Introduction. Concerning the perturbation of the spectrum of a self-adjoint operator, the well-known theorem of Weyl-von Neumann[1] states that *any self-adjoint operator in a separable Hilbert space can be changed into one with a pure point spectrum by the addition of a suitable completely continuous, self-adjoint operator with arbitrarily small Schmidt norm.*

This theorem is no longer true if "Schmidt norm" is replaced by "trace norm". This is a direct consequence of a recent result of Kato,[2] according to which *the absolutely continuous part of the spectrum of a self-adjoint operator is never changed by the addition of a self-adjoint operator with finite trace norm.*

There still remains a gap between these two results, for there are plenty of classes of completely continuous operators other than the Schmidt class and the trace class. These classes are most conveniently described in terms of the *cross norm* introduced by von Neumann and Schatten.[3] The purpose of the present note is to fill in the gap by showing that the theorem of Weyl-von Neumann is true for all unitarily invariant cross norms with the single exception of the trace norm (or its equivalent). This shows at the same time that the trace class is the only allowable class in Kato's theorem, as long as we are concerned with classes defined in terms of unitarily invariant cross norms.

We give a brief exposition of the properties of cross norms needed in the sequel.[4] Let $\mathfrak{H}$ be a separable Hilbert space, $\mathbf{B}$ the space of all bounded linear operators on $\mathfrak{H}$ to $\mathfrak{H}$, $\mathbf{S}\subset\mathbf{B}$ the Schmidt class, $\mathbf{T}\subset\mathbf{S}$ the trace class and $\mathbf{F}\subset\mathbf{T}$ the space of all operators of finite rank. We denote by $\|\ \|$ the ordinary norm, by $\|\ \|_2$ the Schmidt norm and by $\|\ \|_1$ the trace norm. In conformity with Schatten's terminology,[3] a norm $\alpha(X)$ defined on $\mathbf{F}$ will be called a *unitarily*

1) J. von Neumann: Charakterisierung des Spektrums eines Integraloperators, Actualités Sci. Ind., **229**, Paris (1935).

2) T. Kato: Perturbation of continuous spectra by trace class operators, Proc. Japan Acad., **33**, 260–264 (1957). Here the result is obtained in a general, not necessarily separable, Hilbert space.

3) R. Schatten: A theory of cross spaces, Ann. Math. Studies, Princeton (1950).

4) Detailed results concerning unitarily invariant cross norms can be found in Schatten's work cited above.