35. On Countable-Dimensional Spaces

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(Comm. by K. KUNUGI, M.J.A., March 12, 1958)

As is well known, not every infinite-dimensional metric space is the countable sum of zero-dimensional spaces; in fact the Hilbert-cube I_ω is not the countable sum of 0-dimensional spaces. It is known that by the generalized decomposition-theorem due to M. Katětov [1] and to K. Morita [2] a metric space is the countable sum of 0-dimensional spaces if and only if it is the countable sum of finite-dimensional spaces. We call such a space a *countable-dimensional space.* It seems, however, that our knowledge of countable-dimensional spaces is, because of peculiar difficulties to the infinite-dimensional case, very little if compared to that of finite-dimensional spaces.

The purpose of this note is to extend the theory of finite-dimensional spaces to the countable-dimensional case.[1]

All spaces considered in the present note will be assumed to be metric spaces unless the contrary is explicitly stated. $\operatorname{Dim} R$ denotes the Lebesgue dimension of R.

We denote by $\operatorname{order}_p \mathfrak{U}$ for a point p and for a covering $\mathfrak{U}$ of a space R the largest integer n such that there exist n members of $\mathfrak{U}$ which contain p. We also use the notation $B(\mathfrak{U})=\{B(U)\mid U\in\mathfrak{U}\}$, where $B(U)$ means the boundary of U.

Lemma 1. *Let A_n, $n=1,2,\cdots$ be a countable number of 0-dimensional sets of a space R. Let $\{U_\alpha\mid\alpha<\tau\}$[2] be a collection of open sets and $\{F_\alpha\mid\alpha<\tau\}$ a collection of closed sets such that $F_\alpha\subset U_\alpha$, $\alpha<\tau$ and such that $\{U_\beta\mid\beta<\alpha\}$ is locally finite for every $\alpha<\tau$. Then there exists a collection of open sets V_α, $\alpha<\tau$ such that*

1) $F_\alpha\subset V_\alpha\subset U_\alpha$, $\alpha<\tau$,

2) $\operatorname{order}_p B(\mathfrak{V})\leqq n-1$ *for every* $p\in A_n$,

where $\mathfrak{V}=\{V_\alpha\mid\alpha<\tau\}$.

Proof. We shall define, by induction with respect to α, satisfying 1) and

$2)_\alpha$ $\operatorname{order}_p B(\mathfrak{V}_\alpha)\leqq n-1$ for every $p\in A_n$, where $\mathfrak{V}_\alpha=\{V_\beta\mid\beta\leqq\alpha\}$.

We take open sets G_1, W_1 such that

$$G_1\supset F_1,\quad W_1\supset U_1^c,\quad \overline{G}_1\frown\overline{W}_1=\phi.$$

Since A_1 is 0-dimensional, there exists an open, closed set N_1 of A_1 satisfying $\overline{G}_1\frown A_1\subset N_1\subset(\overline{W}_1)^c\frown A_1$. If we put $B_1=N_1\smile F_1$, $C_1=(A_1-N_1)$

1) The detail of the content of this note will be published in an another place.

2) We denote by α,β,γ,τ ordinal numbers.