117. *On Ideals in Semiring*

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Recently, G. Thierrin [6, 7] has discussed some kinds of ideals in any ring and associative (i.e. semigroup). An associative means an algebraic system with an associative law, i.e. semigroup. This terminology was suggested by Dr. F. Klein in his letter (May 15, 1958) to the present writer. In his papers [1, 2], F. Klein has stated that the word is better than semigroup. L. Lesieur and R. Croisot [3-5] have developed a new unified theory of ideals in ring, associative and module. In this paper, we shall consider ideals in a semiring. For fundamental notions on a semiring and its related subjects, see H. S. Vandiver and M. W. Weaver [8].

An ideal P in a semiring is called *completely prime,* if $ab \in P$ implies $a \in P$ or $b \in P$.

An ideal M is called *completely semi-prime,* if $a^2 \in M$ implies $a \in M$.

Following G. Thierrin [6, 7], we shall define a *compressed ideal* in a semiring R. An ideal M is *compressed,* if and only if $a_1^2 a_2^2 \cdots a_n^2 \in M$ for any n implies $a_1 a_2 \cdots a_n \in M$.

Every completely prime ideal is compressed, and every compressed ideal is completely semi-prime.

Theorem 1. *If an ideal M is compressed, then $a_1 a_2 \cdots a_n \in M$ implies $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$ for any positive integers $l_1, l_2, \cdots, l_n$, and $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$ implies $a_1 a_2 \cdots a_n \in M$.*

Proof. $a_1 a_2 \cdots a_n \in M$ implies $a_1^{l_1} a_2 \cdots a_n \in M$. By the remark above, M is completely prime, and we have $a_2 \cdots a_n a_1^{l_1} \in M$.[1] By the same argument, we have $a_2^{l_2} a_3 \cdots a_1^{l_1} \in M$. Hence we have $a_3 \cdots a_n a_1^{l_1} a_2^{l_2} \in M$. This implies $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$.

Conversely, let $a_1^{l_1} a_2^{l_2} \cdots a_n^{l_n} \in M$, then we have $a_1^{l} a_2^{l} \cdots a_n^{l} \in M$ for sufficiently large integer $n > 2$. Therefore, we have $a_1^{2(l-1)} a_2^{2(l-1)} \cdots a_n^{2(l-1)} \in M$. Since M is compressed, then we have $a_1^{l-1} a_2^{l-1} \cdots a_n^{l-1} \in M$. By repeating the same processes, we have $a_1 a_2 \cdots a_n \in M$.

Let M be an ideal in R. We shall call an element x T-element for M if it is $x = x_1 \cdot x_2 \cdots x_n$ such that $x_1^2 x_2^2 \cdots x_n^2 \in M$ for some n. Let us denote by $T^1(M)$ the set of all T-elements for M, and let $T_1(M)$ be the ideal generated by $T^1(M)$. By the induction, we shall $T_n(M)$ as follows: $T_n(M) = T_1(T_{n-1}(M))$ $(n > 1)$. Then each $T_n(M)$ is an ideal and $T_n(M) \subseteq T_{n+1}(M)$. The Thierrin radical $T^*(M)$ of M is

1) See K. Iséki: Ideals in semirings, Proc. Japan Acad., **34**, 29-31 (1958).