131. A Linear Representation of a Countably Infinite Group

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(Comm. by K. KUNUGI, M.J.A., Nov. 12, 1958)

1. Let $\mathcal{G}$ be a countably infinite group and $\mathcal{H}$ the Hilbert space of all complex-valued functions $g \to f(g)$ such that $\sum_{g \in \mathcal{G}} |f(g)|^2$ is finite. For each $g \in \mathcal{G}$, let U_g be the unitary operator on $\mathcal{H}$ defined by $[U_g f](g') = f(g'g)$ and let $M(\mathcal{G})$ the ring of operators generated by $\{U_g\}_{g \in \mathcal{G}}$. Murray and von Neumann have shown that $M(\mathcal{G})$ is a factor of type II_1 if all non-trivial conjugate classes of $\mathcal{G}$ are infinite, and further proposed to expand an arbitrary countably infinite group to a group which has the above property. These results can also be interpreted in the following way: An arbitrary countably infinite group admits a faithful representation on a group of inner automorphisms of a factor of the case (II_1) on a separable Hilbert space.

The object of the present paper is to show the following

Theorem. *Let G be an arbitrary countable group, then G is isomorphic to a group of outer automorphisms of the approximately finite factor on a separable Hilbert space.*

By an automorphism of a factor, we understand a $*$-automorphism, and by a group of outer automorphisms of a factor, we understand a group of automorphisms in which all but the unit element are outer. In proving our theorem, it is sufficient to show the case where G is countably infinite. Indeed, let G be a finite group. Then, for any countably infinite group G' (for example the additive group of integers), the direct product $G \times G'$ is countably infinite and G is embedded isomorphically into $G \times G'$.

The restriction that G is countably infinite is not essential. For an arbitrary group, such a representation will probably be possible, because it will probably be represented as a group of outer automorphisms of a generalized approximately finite factor on an arbitrary (not necessarily separable) Hilbert space. Only for the sake of the simplicity, we confine a group G to be countable.

Noting that approximately finite factors on a separable Hilbert space are all $*$-isomorphic to each other [2], our theorem yields that *an approximately finite factor on a separable Hilbert space has a group of outer automorphisms isomorphic to an arbitrary countable group.* Actually, this note arose from the investigation of the crossed products of rings of operators.[1]

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1) Cf. N. Suzuki: Crossed products of rings of operators, to appear.