131. *On Some Properties of Intermediate Logics*

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In [1] I investigated inclusion and non-inclusion between certain intermediate predicate logics. The purpose of this note is to prove some properties of intermediate logics. We use in this note the same notations as in [1] without definitions.

1. Interpretation of classical logic. THEOREM 1. *LK° and LMK° are minimal in the set of all predicate logics which have the properties* (I) *and* (II) *respectively.*

(I) *For any K-provable sequent $\Gamma\to E$, the sequent $\neg\neg\Gamma\to\neg\neg E$ is provable.*

(II) *For any K-provable sequent $\Gamma\to\Delta$, the sequent $\neg\neg\Gamma\to\neg\neg\Delta$ is provable.*

PROOF. First we prove that (I) and (II) hold in LK° and LMK° respectively. For (I) we use as a deductive system of K-provable sequents the rules of inference in Gentzen's LJ [2] with the axiom schemes $A\to A$ and $\to A\vee\neg A$. As for $A\to A$ and $\to A\vee\neg A$ (I) clearly holds. Let us assume that (I) holds for the upper sequent(s) of any rule of inference. This is proved by an inductive method. As an example, we treat $\to\forall$. From $\neg\neg\Gamma\to\neg\neg A(a)$ we obtain $\neg\neg\Gamma\to\forall x\neg\neg A(x)$. Since $\forall x\neg\neg A(x)\to\neg\neg\forall xA(x)$ is provable in LK°, we obtain $\neg\neg\Gamma\to\neg\neg\forall xA(x)$, which shows that (I) holds for the lower sequent of $\to\forall$.

For (II) we use Gentzen's LK as a deductive system of K-provable $\Gamma\to\Delta$. Only $\to\neg$ and $\to\forall$ are the rules of inference which use MK° in a proof of LMK°. We prove (II) only for $\to\forall$. From $\neg\neg\Gamma\to\neg\neg\Delta$, $\neg\neg A(a)$ we obtain $\neg A(a)$, $\neg\neg\Gamma\to\neg\neg\Delta$ and hence $\exists x\neg A(x)$, $\neg\neg\Gamma\to\neg\neg\Delta$. Thence $\neg\neg\exists x\neg A(x)$, $\neg\neg\Gamma\to\neg\neg\Delta$ is provable. Hence, by applying a cut with this sequent and MK° as the upper sequents of the cut, we obtain $\neg\neg\Gamma\to\neg\neg\Delta$, $\neg\neg\forall xA(x)$.

Secondly, let us assume that LZ and LY have the properties (I) and (II) respectively. Since $\to\forall x(A(x)\vee\neg A(x))$ is K-provable, $\to\neg\neg\forall x(A(x)\vee\neg A(x))$ is Z-provable and hence $LZ\supseteq LK^\circ$. In the same way we see that $\to\neg\neg\forall xA(x)$, $\neg\neg\exists x\neg A(x)$ is Y-provable and hence $LY\supseteq LMK^\circ$ since $\to\forall xA(x)$, $\exists x\neg A(x)$ is K-provable.

2. Decomposition of sequent scheme. We mean by $A\leftrightarrow B$ that both $A\to B$ and $B\to A$. In LJ' the following equivalences are provable.