2. On Ideals Defining Non-Singular Algebraic Varieties

By Shizuo ENDO

(Comm. by Z. SUETUNA, M.J.A., Jan. 12, 1960)

The purpose of this note is to prove the following

Theorem 1. *Let V be a non-singular irreducible algebraic variety of dimension d, defined over a field K in an affine space A^n. Then the ideal defining V over K is generated by at most $(n-d)(d+1)+1$ elements.*

To simplify our expression, we shall denote with $\mathrm{N}_R(\mathfrak{a})$ the minimum number of elements generating the ideal $\mathfrak{a}$ in a Noetherian ring R. Our theorem means $\mathrm{N}_R(\mathfrak{p})\leq(n-d)(d+1)+1$, when R is the polynomial ring of n variables $K[X_1, X_2,\cdots, X_n]$ over K and $\mathfrak{p}$ is the prime ideal defining V over K. Now R is a regular domain as defined e.g. in my former paper [2] and $R/\mathfrak{p}$ becomes also a regular domain as $\mathfrak{p}$ defines a non-singular variety. The rank of $\mathfrak{p}$ is $n-d$, as V is d dimensional (cf. [1]). So our Theorem 1 is contained in the following more general

Theorem 2. *Let R be a regular ring of dimension d and $\mathfrak{p}$ be a prime ideal of rank s in R such that $R/\mathfrak{p}$ is also a regular ring. Then* $\mathrm{N}_R(\mathfrak{p})\leq s(d-s+1)+1$.

We shall begin with some lemmas.

Lemma 1. *Let R be a semi-local ring with maximal ideals $\mathfrak{m}_1, \mathfrak{m}_2,\cdots, \mathfrak{m}_s$ and $\mathfrak{a}=(a_1, a_2,\cdots, a_s)$ be any ideal generated by s elements in R. Then the simultaneous equations $x\equiv a_i$ (mod. $\mathfrak{a}\mathfrak{m}_i$) $1\leq i\leq s$, have a solution in R.*

Proof. Since R is semi-local, we have $R=(\bigcap_{j\neq i}\mathfrak{m}_j, \mathfrak{m}_i)$ for any i. So there exist elements e_i, d_i for any i such that $1=e_i+d_i$, where $e_i\notin\mathfrak{m}_i$, $\in\bigcap_{j\neq i}\mathfrak{m}_j$ and $d_i\in\mathfrak{m}_i$. Then, if we set $a=\sum_{i=1}^{s} e_i a_i$, this is a solution as is required.

Lemma 2. *Let R be a local ring with a maximal ideal $\mathfrak{m}$, $\mathfrak{a}$ be any ideal of R and $\mathfrak{b}$ be an ideal of R contained in $\mathfrak{a}$. Then, if $\mathfrak{a}=\mathfrak{b}+\mathfrak{a}\mathfrak{m}$, we have $\mathfrak{a}=\mathfrak{b}$.*

Proof. Set $\bar{R}=R/\mathfrak{b}$, $\bar{\mathfrak{a}}=\mathfrak{a}/\mathfrak{b}$ and $\bar{\mathfrak{m}}=\mathfrak{m}/\mathfrak{b}$. Then, by our assumption, we have $\bar{\mathfrak{m}}\bar{\mathfrak{a}}=\bar{\mathfrak{a}}$. So, $\bar{\mathfrak{a}}\subset\bigcap_{k=1}^{\infty}\bar{\mathfrak{m}}^k$. Since $\bar{R}$ is a local ring, we have, by Krull's theorem, $\bigcap_{k=1}^{\infty}\bar{\mathfrak{m}}^k=(0)$. That is, $\bar{\mathfrak{a}}=(0)$. This shows $\mathfrak{a}=\mathfrak{b}$.

Lemma 3. *Let R be a Noetherian ring and $\mathfrak{a}, \mathfrak{b}$ be two ideals of R such that $\mathfrak{a}\supset\mathfrak{b}$. If $\mathfrak{a}R_{\mathfrak{m}}=\mathfrak{b}R_{\mathfrak{m}}$ for any maximal ideal $\mathfrak{m}$ of R, we have $\mathfrak{a}=\mathfrak{b}$.*