

31. Commutativity of Some Continuous Magnitude

By Kiyoshi ISÉKI and Hajime SUGITA

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A *magnitude* M is a set of elements $a, b, c, \dots$ and a binary operation called *sum* satisfying the following Conditions I-IV.

I. For every pair a, b of M , the sum $a+b$ exists.

II. For every a, b, c , $(a+b)+c=a+(b+c)$.

Definition. If $a, b \in M$, then $a < b$ means that there is an element x such that $a+x=b$. $a > b$ means $b < a$.

III. $a, b < a+b$ for every a, b .

IV. If a, b are distinct, then either $a < b$ or $b < a$.

Therefore, any magnitude M is linear ordered set.

A magnitude M is *Archimedean*, if $a > b$, then there is a positive integer n such that $nb > a$.

Since O. Hölder, some mathematicians, R. Baer, H. Cartan, F. Loonstra and F. A. Behrend [1], have proved that *any Archimedean magnitude is commutative*: $a+b=b+a$ for every element a, b (see H. G. Forder [2]).

A magnitude M is *continuous*, if every bounded subset has the least upper bound.

In this Note, we shall prove the following

Theorem. *Any continuous magnitude is commutative.*

To prove it, we shall show that a continuous magnitude is Archimedean.

Remark. For the proof, Condition III is essential.

Proof. Suppose that M is not Archimedean, then there are two elements a, b such that $a < b$ and $na \leq b$ for $n=1, 2, \dots$.

Consider the set of elements na ($n=1, 2, \dots$), then the set is bounded, and has a least upper bound c such that $na \leq c$ for all $n=1, 2, \dots$ and for $c' < c$, there is an integer m such that $ma > c'$. From $a < c$, there is an element x such that $a+x=c$. By Condition II, we have $x < a+x=c$. Therefore there is an integer m such that $ma > x$. Hence we have $a+ma > a+x$. (This is proved without commutative law.) This shows $(m+1)a > c$, which is a contradiction. Therefore we complete the proof.

References

- [1] F. A. Behrend: A system of independent axioms for magnitudes, *Journal and Proceedings of Royal Society of New South Wales*, **87**, 27-30 (1953).
- [2] H. G. Forder: *The Foundations of Euclidean Geometry*, New York (1927).