# 48. *On Quasi-normed Spaces. III*

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In this paper, we consider the inverse of a linear transformation of a $(QN)$ space into a $(QN)$ space. Here, we consider a linear transformation $T$ whose domain is a $(QN)$ space $E$ with the power $r$ $(0<r\leq 1)$ and range is a $(QN)$ space $F$ with the power $s$ $(0<s\leq 1)$, see [2], [3] or [4].

If a linear transformation $T$ is one-to-one, then $T$ has the inverse transformation $T^{-1}$ of $F$ onto $E$.

Theorem 1. *A linear transformation $T$ has a bounded inverse if and only if there exists a positive number $m$ such that $\|T(x)\|_s \geq m\|x\|_r^{\frac{s}{r}}$ for all $x\in E$.*

Proof. Suppose that $T$ has a bounded inverse $T^{-1}$, then there exists $M$ such that $\|T^{-1}(y)\|_r\leq M\|y\|_s^{\frac{r}{s}}$, and there exists $x\in E$ such that $y=T(x)$. Therefore,

$$\|T^{-1}(T(x))\|_r\leq M\|T(x)\|_s^{\frac{r}{s}},$$

$$\|x\|_r\leq M\|T(x)\|_s^{\frac{r}{s}}$$

and

$$\|x\|_r^{\frac{s}{r}}\leq M^{\frac{s}{r}}\|T(x)\|_s.$$

Let $M^{\frac{s}{r}}=m^{-1}$, then we have $m\|x\|_r^{\frac{s}{r}}\leq\|T(x)\|_s$.

To prove the inverse, let $\|T(x)\|_s=0$, then $T(x)=0$ and $x=0$. On the other hand $x=0$ implies $m\|x\|_r^{\frac{s}{r}}=0$. Therefore $T$ is one-to-one and has the inverse $T^{-1}$ of $T$.

In Theorem 1, we can take $m$ as the norm $\|T\|_s$ of the transformation, i.e. $\|T(x)\|_s\geq\|T\|_s\|x\|_r^{\frac{s}{r}}$. Consequently, the norm of inverse transformation is defined by $\|T^{-1}\|_r=\|T\|_s^{-\frac{r}{s}}$, hence we have $\|T^{-1}\|_r^s=\|T\|_s^{-r}$.

Now, we shall show that a well-known Banach theorem on inverse transformation is also true for the case of $(QN)$ spaces. First, we shall prove Lemmata.

Lemma 1. *Let $T$ be a bounded linear transformation of $E$ into $F$. If the image under $T$ of the unit sphere $S_1$ in $E$ is dense in some sphere $U_r$ about the origin of $F$, then $T(S_1)$ includes $U_r$.*

Proof. By the assumption, the set $A=U_r\frown T(S_1)$ is dense in $U_r$. Let $y$ be any point of $U_r$. For any $\delta>0$, we take $y_0=0$ and choose inductively a sequence $y_n\in F$ such that $y_{n+1}-y_n\in\delta^n A$ and $\|y_{n+1}-y_n\|$