57. On Two Properties of the Curvature of Continuous Parametric Curves

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1. **Introduction.** In the present continuation of his recent papers [1] to [5] the author proceeds to establish two noteworthy properties of the curvature of continuous parametric curves. They generalize certain well-known results in classical differential geometry of curves.

Let $\boldsymbol{R}^m$ be a Euclidean space of any dimension $m \geqq 2$ throughout this note. Let us consider in this space a parametric curve $\varphi(t)$ of the class C^2, defined and regular on the real line $\boldsymbol{R}$. In other words, we suppose that the coordinate-functions $x_i(t)$ of φ are all twice continuously differentiable $(i=1, 2, \cdots, m)$ and that, furthermore, the derivative of φ, given by $\varphi'(t)=\langle x_1'(t), \cdots, x_m'(t)\rangle$ for all t, never vanishes. Let $s(t)$ denote a length-function for the curve φ, so that $s(t)$ increases strictly and for every closed interval $[a, b]$ the arc-length of φ over $[a, b]$ is equal to the increment $s(b)-s(a)$. We write further $\gamma(t)$ for the spheric representation of φ, given by $\gamma(t)=|\varphi'(t)|^{-1}\varphi'(t)$ for each t. Then everybody knows that the curvature of φ at any point t of $\boldsymbol{R}$ is expressed by the absolute value of the s-derivative $(s)\gamma'(t)$ of the curve γ. Indeed this is often adopted as the definition of curvature.

Now the extension of this statement to curves more general than φ considered above is the concern of our first theorem (§3). It should be noted that in our paper [4] we defined curvature in a way different from the aforesaid standard definition and that therefore the propounded extension is not a definition but a theorem requiring a regular proof. As for our second theorem (§5), we must omit the explanation of its origin owing to space limitation.

2. **Direction-curves.** Consider in $\boldsymbol{R}^m$ a continuous light curve $\varphi(t)=\langle x_1(t), \cdots, x_m(t)\rangle$, defined on $\boldsymbol{R}$ and locally straightenable (see [4] §2). Then φ is necessarily locally rectifiable by [1] §64. As in [4], the length and bend of φ over an interval I (of any type) will be denoted by $S(I)$ and $\Omega(I)$ respectively, and the induced measure-length and measure-bend by S_* and Ω_* respectively. Further, we shall continue using the symbol $\rho(t)$ of [4] to denote the curvature (in our sense) of φ at a point t of $\boldsymbol{R}$. We remark in passing that, as easily seen, the definition of bend adopted in [4] is compatible