144. Cohomology Mod 2 of the Compact Exceptional Group E_8

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1. By $\boldsymbol{E}_8$ we mean a compact exceptional Lie group whose local structure is usually expressed by the same letter. It is unique up to isomorphisms. In this note we determine the cohomology ring mod 2 of $\boldsymbol{E}_8$. The result is as follows:

Theorem 1. $H^*(\boldsymbol{E}_8; Z_2)=Z_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4)$
$$\otimes \Lambda_2(x_{17}, x_{23}, x_{27}, x_{29})$$
where the suffix of each generator denotes its degree.

Thus $H^*(\boldsymbol{E}_8; Z_2)$ is a truncated polynomial ring over Z_2 with 8 generators of degrees 3, 5, 9, 15, 17, 23, 27 and 29 respectively, and the heights of generators are 16, 8, 4, 4, 2, 2, 2, 2.

The relations among generators by Steenrod squares are as follows:

Theorem 2. *In the above theorem, the generators of* $H^*(\boldsymbol{E}_8; Z_2)$ *can be chosen to satisfy the relations*:
$$x_5=Sq^2x_3,\ x_9=Sq^4x_5,\ x_{17}=Sq^8x_9,$$
$$x_{23}=Sq^8x_{15},\ x_{27}=Sq^4x_{23},\ x_{29}=Sq^2x_{27}.$$
It is still open whether $Sq^2x_{15}=0$ or x_{17}.

2. Further we obtain the

Proposition 1. $\boldsymbol{E}_7$ *is totally non-homologous zero* mod 2 *in* $\boldsymbol{E}_8$.

In this proposition $\boldsymbol{E}_7$ means the compact simply-connected exceptional group with the local structure expressed by the same letter. This proposition, combined with the prop. 22.4 of Borel [4] and the props. 2.8, 3.12 of [3], proves the

Theorem 3. *In the inclusions* $\boldsymbol{G}_2\subset\boldsymbol{F}_4\subset\boldsymbol{E}_6\subset\boldsymbol{E}_7\subset\boldsymbol{E}_8$, *every subgroup is totally non-homologous zero* mod 2 *in any bigger group containing it, where each exceptional group denotes the compact simply-connected one.*

This theorem holds only for homologies "mod 2".

In [3, 4] the cohomology rings mod 2 of the first four simply-connected exceptional groups are calculated. We shall restate them here.

(1) $H^*(\boldsymbol{G}_2; Z_2)=Z_2[x_3]/(x_3^4)\otimes\Lambda_2(x_5)$,

(2) $H^*(\boldsymbol{F}_4; Z_2)=Z_2[x_3]/(x_3^4)\otimes\Lambda_2(x_5, x_{15}, x_{23})$,

(3) $H^*(\boldsymbol{E}_6; Z_2)=Z_2[x_3]/(x_3^4)\otimes\Lambda_2(x_5, x_9, x_{15}, x_{17}, x_{23})$,

(4) $H^*(\boldsymbol{E}_7; Z_2)=Z_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4)\otimes\Lambda_2(x_{15}, x_{17}, x_{23}, x_{27})$,

where the suffix of each generator denotes the degree. Furthermore