111. A Note on the Extension of Semigroups with Operators

By Takayuki TAMURA and Donald G. BURNELL
University of California, Davis, California, U.S.A.

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In this note we shall report some theorems concerning the theory of extension of semigroups with operators, without detailed proof. By a mono-endomorphism of a semigroup we mean a one-to-one endomorphism of a semigroup. Let S be a semigroup which is not necessarily commutative and suppose that Γ is a commutative semigroup of some mono-endomorphisms α of S, that is, Γ is not necessarily composed of all mono-endomorphisms of S. Let $+$ denote the operation in S and αx the image of an element x under α.

$$\left.\begin{array}{ll}(1.1) & \alpha(x+y)=\alpha x+\alpha y \\ (1.2) & (\alpha\beta)x=(\beta\alpha)x \\ (1.3) & \alpha x=\alpha y \text{ implies } x=y\end{array}\right\} \text{ for } \alpha, \beta\in\Gamma;\ x, y\in S.$$

We shall call such an S a semigroup with Γ denoted by (s, Γ).

Theorem 1. *For* (S, Γ), *there exists* $(\bar{S}, \bar{\Gamma})$ *such that*

(2.1) S *is embedded into* $\bar{S}$,

(2.2) Γ *and* $\bar{\Gamma}$ *are isomorphic*,

(2.3) *Each* $\bar{\alpha}\in\bar{\Gamma}$ *is an extension of* $\alpha\in\Gamma$ *to* $\bar{S}$, *and* $\bar{\alpha}$ *is an automorphism of* $\bar{S}$.

(2.4) $(\bar{S}. \bar{\Gamma})$ *is the smallest extension of* (S, Γ) *in the following meaning*: *If* $(\bar{\bar{S}}, \bar{\bar{\Gamma}})$ *is any extension satisfying* (2.1), (2.2), *and* (2.3), *then* $\bar{S}$ *is embedded into* $\bar{\bar{S}}$.

Proof. Consider the set of all pairs (x, α) of $x\in S$ and $\alpha\in\Gamma$ and we introduce a relation as $(x, \alpha)\sim(y, \beta)$ iff $\beta x=\alpha y$. Then it is an equivalence relation. Let $\overline{(x, \alpha)}$ denote an equivalence class containing (x, α) and let $\bar{S}$ be the set of all equivalence classes. We define an operation in $\bar{S}$ as follows:

$$\overline{(x, \alpha)}+\overline{(y, \beta)}=\overline{(\beta x+\alpha y, \alpha\beta)}.$$

It is shown that this operation is single valued on $\bar{S}$, and $\bar{S}$ is a semigroup into which S is embedded under the mapping $\Sigma: S\ni x\rightarrow \overline{(\alpha x, \alpha)}\in\bar{S}$ where $\overline{(\alpha x, \alpha)}$ is independent of the choice of α. For each α, a mapping $\bar{\alpha}$ of $\bar{S}$ into $\bar{S}$ is defined as follows:

$$\bar{\alpha}\overline{(z, \gamma)}=\overline{(\alpha z, \gamma)}.$$

We can see that this mapping is single-valued on $\bar{S}$ and $\bar{\alpha}$ is a mono-