# 2. *Existence of Pseudo-Analytic Differentials on Riemann Surfaces. II*

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III. Existence theorems. 1. Definition 3.1. *Let $\gamma$ be an analytic closed curve on $R$, and $\omega$ be a real differential of $C$. The integral*

$$\int_\gamma \frac{1}{\sqrt{\sigma}}\omega \tag{3.1}$$

*is called the $\sigma$-period of $\omega$ over $\gamma$, and denoted by $P_\sigma(\omega;\gamma)$. A differential $\omega$ is called $\sigma$-exact if all its $\sigma$-periods vanish.*

The $\sigma$-exact differential can be written as $Du$ for $u\in C^1$.

Theorem 3.1. *Let $\gamma$ be an analytic closed curve which does not devide $R$, then there exists a differential $\omega\in H$ such that $P_\sigma(\omega;\gamma)=1$ and $\sigma$-exact in $R-\gamma$.*

*Proof.* We can construct a closed differential $\eta\in C^2\cap L^2$ which is $\sigma$-exact in $R-\gamma$ and $P_\sigma(\eta:\gamma)=1$. Set $\eta_1=\sqrt{\sigma}\,\eta$, then $D_1\eta_1=0$. Therefore we have $\eta_1=\omega_h+\omega_1$ with $\omega_h\in H$, and $\omega_1\in E$. Since $\eta_1\in C_1$, we have $\omega_1\in C^1$ and therefore $\omega_1=Du$ with $u\in C^2$. In $R-\gamma$, we have $\omega=\eta_1-Du$, and hence $\omega$ is $\sigma$-exact there. Moreover we have

$$\int_\gamma \frac{1}{\sqrt{\sigma}}\omega=\int_\gamma \frac{1}{\sqrt{\sigma}}(\eta_1-Du)=\int_\gamma (\eta-du)=\int_\gamma \eta=1.$$

2. Let $F(p)$ and $G(p)$ be the functions of $C^{1+\alpha}$ satisfying

$$-i\overline{F}G>0 \quad\text{and}\quad M\geqq |F|+|G|\geqq M^{-1}>0. \tag{3.2}$$

An $(F,G)$-pseudo-analytic function is an $[a,b]$-analytic function with

$$a=-\frac{\overline{F}G_{\bar{z}}-F_{\bar{z}}\overline{G}}{F\overline{G}-\overline{F}G},\quad b=\frac{FG_{\bar{z}}-F_{\bar{z}}G}{F\overline{G}-\overline{F}G}, \tag{3.3}$$

and an $(F,G)$-pseudo-analytic differential is an $[a,b]$-analytic differential with

$$a=-\frac{\overline{F}G_{\bar{z}}-F_{\bar{z}}\overline{G}}{F\overline{G}-\overline{F}G},\quad b=-\frac{FG_{z}-F_{z}G}{F\overline{G}-\overline{F}G}. \tag{3.4}$$

Under the condition (3.2), $(F,G)$-analytic function of the 2nd kind $\chi(p)=u(p)+iv(p)$ satisfy the equation

$$\begin{cases} v_x=-\sigma u_y \\ v_y=\sigma u_x \end{cases}\qquad \sigma=i\frac{F}{G}>0. \tag{3.5}$$

Since $\sigma\in C^{1+\alpha}$, $u(p)$ is in $C^2$, and hence $u$ is $\sigma$-harmonic.

3. We fix the point $p_0\in R$, a neighborhood $V$ of $p_0$, and its local parameter $z$. Let $W_0(z)$ be the $(F,G)$-analytic function similar to the function $1/z^n$ $(n\geqq 1)$ in $V$. Let $\chi_0(z)=u_0+iv_0$ be the $(F,G)$-analytic