19. The ε-Entropy of Some Classes of Harmonic Functions

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1. Let K be a bounded continuum in q-dimensional Euclidian space and G be a bounded open set containing K. For complex-valued function $u(x)$ in G, we define $\|u(x)\|=\sup_{x\in K}|u(x)|$. We consider classes $H_G(C)$ of functions $u(x)$ which are harmonic in G and bounded in G by the constant C. When we introduce the metric $\|\cdot\|$ in $H_G(C)$, we shall denote it by $H_G^K(C)$.

The purpose of the present paper is to compute "ε-entropy" and "ε-capacity" of $H_G^K(C)$ for some K and G. The exact formulae for them are given in 3. Using these results, we can compute the "functional dimension" of the vector space of harmonic function in 4.

The problem of computing ε-entropy of the space of solutions of partial differential equations was posed by Prof. H. Yoshizawa.

2. Following [3], we shall list definitions which are necessary to state our results. Let R be a metric space and A a set in R.

DEFINITION 1. A set B in R is called an ε-net for the set A if every points of A is at a distance not exceeding ε from some point of B.

DEFINITION 2. A set B in R is called ε-separated if the distance of any distinct points of B are greater than ε.

Now we assume the set A is totally bounded.

DEFINITION 3. $N(\varepsilon, A)$ is the minimal number of points in all possible ε-net for A. $H(\varepsilon, A)=\log N(\varepsilon, A)$ is called ε-entropy of the set A. ($\log N$ will always denote the logarithm of the number N in the base 2.)

DEFINITION 4. $M(\varepsilon, A)$ is the maximal number of points in all possible ε-separated subsets of the set A. $C(\varepsilon, A)=\log M(\varepsilon, A)$ is called the ε-capacity of A.

We shall state a simple theorem which will be used later [3].

THEOREM 1. $M(2\varepsilon, A)\leqq N(\varepsilon, A)$

3. Our result is as follows.

THEOREM. Let $K_r=\{x; \sum_{i=1}^q x_i^2\leq r^2\}$ and $G_R=\{x; \sum_{i=1}^q x_i^2< R^2\}$ in q-dimensional space. Then

$$H(\varepsilon,\ H_{G_R}^{K_r}(C))=\{4/q!\,(\log R/r)^{q-1}\}\,(\log 1/\varepsilon)^q+O((\log 1/\varepsilon)^{q-1}\log\log 1/\varepsilon),$$
$$C(2\varepsilon, H_{G_R}^{K_r}(C))=\{4/q!\,(\log R/r)^{q-1}\}\,(\log 1/\varepsilon)^q+O((\log 1/\varepsilon)^{q-1}\log\log 1/\varepsilon).$$

(For notations, see 1 and 2.)

REMARK. From Theorem 1 it is sufficient to estimate $H(\varepsilon, A)$ from