63. On a Theorem of Cluster Sets

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1. Let D be an arbitrary domain in the z-plane with boundary Γ and let E be a totally disconnected closed set contained in Γ. Supposing that $w=f(z)$ is non-constant, single-valued and meromorphic in D, we associate with each point $z_0 \in \Gamma$ the following sets of values.

(i) *The cluster set* $C_D(f, z_0)$. $\alpha \in C_D(f, z_0)$ if there exists a sequence of points $\{z_n\}$ with the following properties:

$$z_n \in D, \lim_{n\to\infty} z_n = z_0 \quad \text{and} \quad \lim_{n\to\infty} f(z_n)=\alpha.$$

(ii) *The boundary cluster set* $C_{\Gamma-E}(f, z_0)$. $\alpha \in C_{\Gamma-E}(f, z_0)$ if there exists a sequence of points $\{\zeta_n\}$ of $\Gamma-(\{z_0\}\cup E)$ such that

$$w_n \in C_D(f, \zeta_n) \quad \text{for each} \quad n,$$

$$z_0=\lim_{n\to\infty} \zeta_n \quad \text{and} \quad \alpha=\lim_{n\to\infty} w_n.$$

(iii) *The range of values* $R_D(f, z_0)$. This is the set of values α such that

$$z_n \in D, \lim_{n\to\infty} z_n = z_0 \text{ and } f(z_n)=\alpha \text{ for every } n.$$

In the theory of cluster sets, the following theorem is one of the most important.[1]

THEOREM. *If E is of capacity*[2] *zero and z_0 is a point of E such that $U(z_0)\cap(\Gamma-E)\neq\phi$ for every neighborhood $U(z_0)$ of z_0, then the set*

$$\Omega=C_D(f, z_0)-C_{\Gamma-E}(f, z_0)$$

is empty or open.

In the case where D is the unit disc $|z|<1$, we can replace $C_{\Gamma-E}(f, z_0)$ in this theorem by a considerably smaller set and obtain yet the same assertion (see Ohtsuka [5] and Noshiro [3]).[1] We shall show in the below that, in the general case where D is an arbitrary domain, we can also replace $C_{\Gamma-E}(f, z_0)$ by a considerably smaller set to obtain the same assertion of the theorem.

2. We now define new sets of values.

(iv) *The cross cluster set* $C_D^{\oplus}(f, z_0)$. $\alpha \in C_D^{\oplus}(f, z_0)$ if there exists a sequence of points $\{z_n\}$ with the following properties:

$$z_n \in D,\ \Re z_n = \Re z_0 \text{ or } \Im z_n = \Im z_0 \text{ for each } n,^{3)}$$

$$\lim_{n\to\infty} z_n = z_0 \quad \text{and} \quad \lim_{n\to\infty} f(z_n)=\alpha.$$

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1) Cf. Noshiro [4].

2) In this note, capacity means always logarithmic capacity.

3) For a complex number z, we denote by $\Re z$ and $\Im z$ the real and the imaginary part of z respectively.