59. On Bochner Transforms

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1. If $\varphi(r)$ is a function of the distance r from the origin of h-dimensional Euclidean space, then the Fourier transform of $\varphi(r)$, that is the integral transform by the kernel function $\exp(2\pi_i(x_1y_1+ +\cdots+x_hy_h))$, is also a function of r and is expressed as following:

$$T\varphi(r)=2\pi r^{1-\frac{h}{2}}\int_0^\infty J_{\frac{h}{2}-1}(2\pi r\rho)\rho^{\frac{h}{2}}\varphi(\rho)\,d\rho,$$

where $J_\nu(x)$ is Bessel function ([1] p. 69, Theorem 40).

By the general theory of Fourier transform the linear operator T has the properties:

(a) *T transforms $\varphi(ur)$ to $|u|^{-h}T\varphi\left(\frac{r}{u}\right)$ if $u\neq 0$,*

(b) *T transforms $e^{-\pi r^2}$ to $e^{-\pi r^2}$,*

(c) *there exists a number series $\{a_0, a_1,\cdots\}$ which satisfies*
$\sum_{n=0}^\infty a_n\varphi(\sqrt{n})=\sum_{n=0}^\infty a_nT\varphi(\sqrt{n})$ (Poisson summation formula),

(d) $T^2\varphi=\varphi$ (the inversion formula),

and

(e) $\int_0^\infty |T\varphi(r)|^2r^{k-1}\,dr=\int_0^\infty |\varphi(r)|^2r^{k-1}\,dr$ (Parseval formula).

In his paper [2] Bochner proved that the properties (a), (b) and (c) characterize the operator T.

We shall describe here the theorem of Bochner in somewhat modified form:

Let us denote by $\mathfrak{P}_0$ the family of all functions $\varphi(x)$ on $[0, \infty)$ such that $\left(\frac{d}{xdx}\right)^r\varphi(x)$ exists at 0 for any r and every derivative of $\varphi(x)$ decreases rapidly as x tends to infinity.

Theorem of Bochner. *Let T be a linear operator from $\mathfrak{P}_0$ to $\mathfrak{P}_0$ which satisfies the following conditions:*

(A) *T transforms $\varphi(ux)$ to $|u|^{-h}T\varphi\left(\frac{x}{u}\right)$ for any $u\neq 0$, where h is a positive constant,*

(B) *$e^{-\frac{2\pi}{\lambda}x^2}$ is an eigenfunction of T, where λ is a positive constant, and*

(C) *there exists number series $\{a_0, a_1, a_2,\cdots\}$ such that $\sum_{n=1}^\infty\frac{|a_n|}{n^{s_0}}$ converges for a positive number s_0 and*